MATH 223 Spring 2025
Assignment 13
Due: Friday, March 14

Reading

Read carefully Section 4.6.1 "Economic Growth" and Section 5.1 "Differentiating Compositions of Functions" in our text *Multivariable Calculus: A Linear Algebra Based Approach.*

Writing

Write out solutions of Exercises 30 31, 32 in Chapter 4 (Old 23, 24, 25) and Exercises A and B below.

A. Show that the Mean Value formula of Theorem 4.4.1 (*Generalized Mean Value Theorem)* also takes the form

$$\frac{f(\mathbf{y})-f(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = f_{\mathbf{u}}(\mathbf{x_0})$$

where the unit vector is $\mathbf{u} = (\mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$.

B. The Mean Value Theorem may fail for vector-valued functions. Consider the function

$$f(t) = (\sin t, \sin 2t), 0 \leq t \leq \pi.$$

Show that there is no point x_0 between x and y at which

$$\frac{f(y)-f(x)}{|y-x|} = f'(x_0)$$ where $x = 0$ and $y = \pi$.

Here is an optional, extra credit problem:

C. The fundamental derivative approximation $f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})\mathbf{h}$ for differentiable real-valued functions $f\colon \mathbb{R}^n \rightarrow \mathbb{R}$ is the first degree portion of higher degree approximations. We define the ***Nth-degree Taylor Approximation*** to $f(\mathbf{x}+\mathbf{h})$ by

$$f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})\mathbf{h} + \frac{1}{2!} f_{\mathbf{hh}}(\mathbf{x}) + \frac{1}{3!} f_{\mathbf{hhh}}(\mathbf{x}) + \cdots + \frac{1}{N!} f_{\mathbf{hhhh\ldots hh}}(\mathbf{x})$$

where we assume that the required Nth-order derivatives are continuous on an open neighborhood of **x**.

We define the higher-order derivatives with respect to the vector **h** recursively by generalizing ordinary higher-order partials in this way:

$$f_{\mathbf{hh}}(\mathbf{x}) = \frac{\partial^2 f}{\partial \mathbf{h}}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{h}}\left(\frac{\partial f}{\partial \mathbf{h}}\right)(\mathbf{x})$$

$$f_{\mathbf{hhh}}(\mathbf{x}) = \frac{\partial^3 f}{\partial \mathbf{h}}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{h}}\left(\frac{\partial^2 f}{\partial \mathbf{h}^2}\right)(\mathbf{x})$$

$$\cdots$$

$$f_{\mathbf{hhhh..h}}(\mathbf{x}) = \frac{\partial^N f}{\partial \mathbf{h}}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{h}}\left(\frac{\partial^{N-1} f}{\partial \mathbf{h}^{N-1}}\right)(\mathbf{x})$$

Let $f(x,y) = \sin(x^2+y)$, $\mathbf{x} = (0,0)$, and $\mathbf{h} = (h,k)$. Compute the second-degree Taylor approximation to $f(h,k)$