

MATH 223: Multivariable Calculus

Projectile Motion

Horizontal

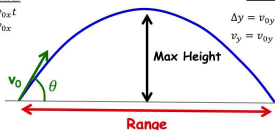
$$\Delta x = v_{0x}t$$

$$v_x = v_{0x}$$

Vertical

$$\Delta y = v_{0y}t - \frac{1}{2}gt^2$$

$$v_y = v_{0y} - gt$$



February 17, 2025



Notes on Assignment 3
Assignment 4

Review: Curves and Tangent Lines

Example 1: $\mathbf{F}(x) = (x^3 + 7x + 3, 8 + \sin x), -2 \leq x \leq 2.$

$$\mathbf{F}(0) = (3, 8)$$

$$\text{So } \mathbf{F}'(x) = (3x^2 + 7, \cos x)$$

$$\text{implying } \mathbf{F}'(0) = (7, 1)$$

Tangent Line

$$\begin{aligned}\mathbf{L}(t) &= \mathbf{F}(0) + t \mathbf{F}'(0) \\ &= (3, 8) + (7, 1)t \\ &= (3 + 7t, 8 + t)\end{aligned}$$

Review: Curves and Tangent Lines

Example 2:

$$\mathbf{F}(x) = (x^3 + 7x + 3, 8 + \sin x, \ln 1 + x^2), -2 \leq x \leq 2.$$

$$\mathbf{F}(0) = (3, 8, 0)$$

$$\text{So } \mathbf{F}'(x) = (3x^2 + 7, \cos x, \frac{2x}{1+x^2})$$

$$\text{implying } \mathbf{F}'(0) = (7, 1, 0)$$

Tangent Line

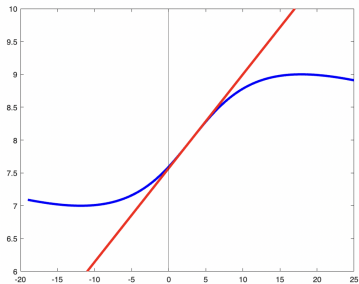
$$\mathbf{L}(t) = \mathbf{F}(0) + t\mathbf{F}'(0)$$

$$= (3, 8, 0) + (7, 1, 0)$$

$$= (3 + 7t, 8 + t, 0)$$

Plot Curve and Tangent Line Together

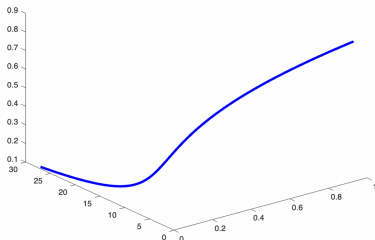
```
hold on
x = [-2:.02:.2];
plot(x.^3+.7*x+.3, .8+.sin(x), 'b', 'LineWidth', 3);
plot(3+.7*x, .8+.x, 'r', 'LineWidth', 3);
xline(0)
```



Plotting Vector-Valued Functions in MATLAB

$$F(t) = (t^7, t^{-3}, \sin t^2), 1/3 \leq t \leq 1$$

```
t = [1/3 : .01 : 1];  
plot3(t.^7, t.^-3, sin(t.^2), 'b', 'LineWidth', 3)
```



Integrating Vector-Valued Functions of a Real Number

Example 3: Find all vector-valued functions of a real number $\mathbf{p}(t)$ such that

$$\mathbf{p}'(t) = \left(\frac{1}{t^2 + 1}, \frac{t}{t^2 + 1} \right)$$

Solution:

$$\mathbf{p}(t) = \int \mathbf{p}'(t) dt = \int \left(\frac{1}{t^2 + 1}, \frac{t}{t^2 + 1} \right) dt$$

$$= \left(\arctan t, \frac{1}{2} \ln(t^2 + 1) \right) + (C_1, C_2)$$

$$= \left(\arctan t + C_1, \frac{1}{2} \ln(t^2 + 1) + C_2 \right)$$

Example 3 Continued: $\mathbf{p}(t) = (\arctan t + C_1, \frac{1}{2} \ln(t^2 + 1) + C_2)$

(i) Find the particular solution so that $\mathbf{p}(0) = (3, 4)$

$$3 = \arctan 0 + C_1 = 0 + C_1 \text{ so } C_1 = 3$$

$$4 = \frac{1}{2} \ln(0^2 + 1) + C_2 = 0 + C_2 \text{ so } C_2 = 4$$

$$\text{Hence } \mathbf{p}(t) = (\arctan t + 3, \frac{1}{2} \ln(t^2 + 1) + 4)$$

(ii) Find the particular solution so that $\mathbf{p}(1) = (a, b)$

$$a = \arctan 1 + C_1 = \pi/4 + C_1 \text{ so } C_1 = a - \pi/4$$

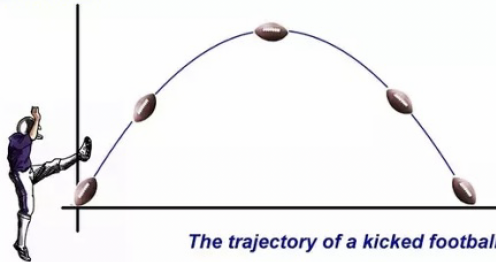
$$b = \frac{1}{2} \ln(t^2 + 1) + C_2 = \frac{1}{2} \ln 2 + C_2 \text{ so } C_2 = a - \pi/4 - \frac{1}{2} \ln 2$$

$$\text{Hence } \mathbf{p}(t) = (\arctan t + a - \pi/4, \frac{1}{2} \ln(t^2 + 1) + a - \pi/4 - \frac{1}{2} \ln 2)$$

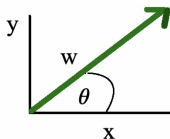
Projectile Motion



Projectile Motion



The trajectory of a kicked football



Initial Velocities: $x'_0 = w \cos \theta, y'_0 = w \sin \theta$
 x and y are function of time t .

$$x'' = 0$$

$$x' = C = w \cos \theta$$

$$x = w \cos \theta t + x_0$$

$$y'' = -g$$

$$y' = -gt + C = -gt + w \sin \theta$$

$$y = -\frac{g}{2}t^2 + (w \sin \theta)t + y_0$$

$$x(t) = w \cos \theta t + x_0, y(t) = -\frac{g}{2}t^2 + (w \sin \theta)t + y_0$$

$$x(t) = w \cos \theta t + x_0$$

$$y(t) = -\frac{g}{2}t^2 + (w \sin \theta)t + y_0$$

Suppose $x_0 = 0, y_0 = 0$

Then

$$x(t) = w \cos \theta t$$

$$y(t) = -\frac{g}{2}t^2 + (w \sin \theta)t$$

Note:

$$t = \frac{x}{w \cos \theta}$$

$$y = -\frac{g}{2} \left(\frac{x^2}{w^2 \cos^2 \theta} \right) + \frac{w \sin \theta}{w \cos \theta} x$$

Graph of y versus x is a Downward Pointing Parabola

$$x(t) = w \cos \theta t$$

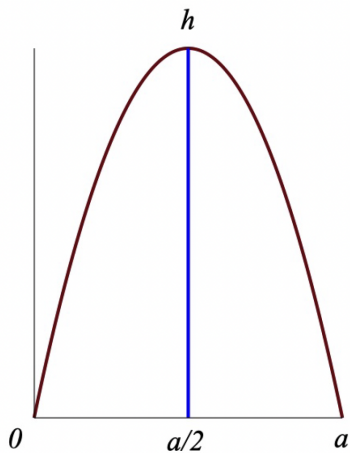
$$y(t) = -\frac{g}{2}t^2 + (w \sin \theta)t = t \left(-\frac{g}{2}t + w \sin \theta\right)$$

Now $y = 0$ at $t = 0$ and when $w \sin \theta = \frac{g}{2}t$
 which occurs when $t = \frac{w \sin \theta}{g}$

At this time

$$x = w \cos \theta \left(\frac{w \sin \theta}{g} \right) = \frac{2w^2 \sin \theta \cos \theta}{g} = \frac{w^2}{g} \sin 2\theta$$

So Maximum Horizontal Distance occurs when $\sin(2\theta) = 1$;
 that is, $\theta = \frac{\pi}{4}$



Maximum Height h

$$\frac{a}{2} = w \cos \theta, h = -\frac{g}{2} t^2 + w \sin \theta t$$

Maximum h occurs when $t = \frac{w \sin \theta}{g}$ where $h'(t) = 0$.

$$\text{At this time, } x = w \cos \theta \frac{w \sin \theta}{g} = \frac{w^2}{g} \sin \theta \cos \theta$$

$$h = -\frac{g}{2}t^2 + w \sin \theta t$$

$$\text{At } t = \frac{w \sin \theta}{g}$$

$$\begin{aligned} h &= -\frac{g}{2} \left(\frac{w \sin \theta}{g} \right)^2 + w \sin \theta \left(\frac{w \sin \theta}{g} \right) \\ &= -\frac{g}{2} \frac{w^2 \sin^2 \theta}{g^2} + \frac{w^2 \sin^2 \theta}{g} \\ &= -\frac{w^2 \sin^2 \theta}{2g} + \frac{w^2 \sin^2 \theta}{g} \\ &= \frac{w^2 \sin^2 \theta}{2g} \end{aligned}$$

$$\text{where } x = \frac{w^2}{g} \sin \theta \cos \theta$$

To have $y = h$ at $x = a/2$:

$$h = \frac{w^2 \sin^2 \theta}{2g}, \frac{a}{2} = \frac{w^2}{g} \sin \theta \cos \theta$$

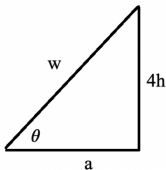
$$\frac{h}{a/2} = \left(\frac{w^2 \sin^2 \theta}{2g} \right) \left(\frac{g}{w^2 \sin \theta \cos \theta} \right)$$

$$\frac{2h}{a} = \frac{\sin \theta}{2 \cos \theta}$$

$$\frac{4h}{a} = \tan \theta$$

$$\theta = \arctan \left(\frac{4h}{a} \right)$$

Recall $h = \frac{w^2 \sin^2 \theta}{2g}$ and $\tan \theta = \frac{4h}{a}$



$$w = \sqrt{16h^2 + a^2} \text{ so } \sin \theta = \frac{4h}{\sqrt{16h^2 + a^2}}$$

$$h = \frac{w^2}{2g} = \frac{w^2}{2g} \left(\frac{16h^2}{16h^2 + a^2} \right) \text{ Solve for } w^2:$$

$$w^2 = \frac{(2gh)(16h^2 + a^2)}{16h^2} = \frac{g(16h^2 + a^2)}{8h}$$

Real-Valued Functions of Vectors

Example: $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + 2y^2} & (x, y) \neq (0, 0) \\ 0 & (0, 0) \end{cases}$$

What happens as (x, y) approaches $(0, 0)$?
Consider approaching along line $y = mx$. Then

$$\frac{xy}{x^2 + 2y^2} = \frac{xmx}{x^2 + 2(m^2x^2)} = \frac{m}{1 + 2m^2}$$