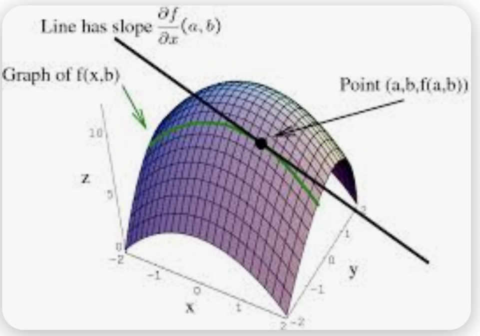
MATH 223: Multivariable Calculus



Class 7: February 24, 2025

Notes on Assignment 6
Assignment 7
Parametric Surfaces in MATLAB

Announcements

Exam 1: Monday, March 3
7 PM – ?

This Week

Partial Derivatives
Parametric Surfaces
Applications

Partial Derivatives

Setting: $f : \mathcal{R}^2 \to \mathcal{R}^1$

Partial with respect to x at (a, b) :

$$\frac{\partial f}{\partial x}(a, b) = f_x(a, b) = \lim_{t \to 0} \frac{f(a+t, b) - f(a, b)}{t}$$

Partial with respect to y at (a, b) :

$$\frac{\partial f}{\partial y}(a, b) = f_y(a, b) = \lim_{t \to 0} \frac{f(a, b+t) - f(a, b)}{t}$$

Note: BOTH OF THESE LIMITS WILL BE NUMBERS

Example: $f(x,y) = x^2y$ at (2,3)
Value of function at (2,30 is $f(2,3) = 2^2 \times 3 = 12$

First $f_x(2,3)$:

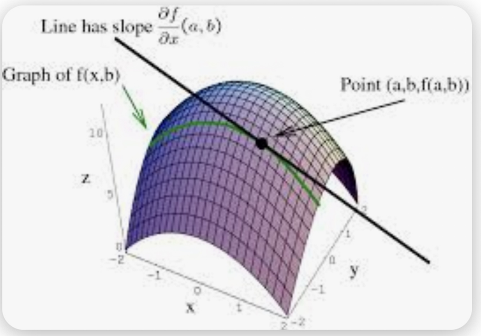
$$\begin{aligned}\frac{f(2+t,3)-f(2,3)}{t} &= \frac{(2+t)^2 \times 3 - 12}{t} = \frac{(4+4t+t^2)3-12}{t} \\ &= \frac{12+12t+3t^2-12}{t} = \frac{12t+3t^2}{t} \\ &= 12+3t \text{ if } t \neq 0\end{aligned}$$

so $f_x(2,3) = \lim_{t\to 0}(12+3t) = 12.$
Now $f_y(2,3)$:

$$\begin{aligned}\frac{f(2,3+t)-f(2,3)}{t} &= \frac{2^2(3+t)-12}{t} = \frac{12=4t-12}{t} \\ &= 4 \text{ if } t \neq 0\end{aligned}$$

so $f_y(2,3) = \lim_{t\to 0} 4 = 4.$

What's Going on Visually?



So Far for $f(x,y) = x^2 y$ at $\mathbf{a} = (2{,}3)$:
$f(\mathbf{a}) = 12, f_x(\mathbf{a}) = 12, f_y(\mathbf{a}) = 4.$

We Can Write the Equation of a Plane:

$$\begin{aligned} z &= 12 + 12(x-2) + 4(y-3) \\ &= f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) \\ &= f(\mathbf{a}) + (12,4) \cdot (x-2, y-3) \\ &= f(\mathbf{a}) + (f_x(\mathbf{a}), f_x(\mathbf{a})) \cdot (\mathbf{x} - \mathbf{a}) \\ &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) \end{aligned}$$

where $\nabla f(\mathbf{a})$ is called the **gradient** of f at $\mathbf{a}$.
Note: We can also write
$z = 12 + 12x - 24 + 4y - 12 = 12x + 4y - 24$
Analogy: Tangent Line for $f : \mathcal{R}^1 \to \mathcal{R}^1$:
$y = f(a) + f'(a)(x-a)$

Calculations

Apply Usual Rules of Differentiation

For f_x : Treat y as a constant For f_y : Treat x as a constant

$$\text{Example: } w = f(x,y,z) = x^2y + y^2 \ln z$$
$$w_x = f_x(x,y,z) = 2xy + 0 = 2xy$$
$$w_y = f_y(x,y,z) = x^2 + 2y \ln z$$
$$w_z = f_z(x,y,z) = 0 + \frac{y^2}{z} = \frac{y^2}{z}$$

Higher Order Partials

$$\begin{matrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{matrix}$$

Example: $f(x,y) = x^2y + \sin x\, e^y$

$$f_x(x,y) = 2xy + \cos x\, e^y$$

yields

$$f_{xx}(x,y) = 2y - \sin x\, e^y$$
$$f_{xy}(x,y) = 2x + \cos x\, e^y$$

$$f_y(x,y) = x^2 + \sin x\, e^y$$

yields

$$f_{yx}(x,y) = 2x + \cos x\, e^y$$
$$f_{yy}(x,y) = \sin x\, e^y.$$

Tangent Plane To Graph of $f : \mathcal{R}^n \to \mathcal{R}^1$ at point $(\mathbf{a}, f(\mathbf{a}))$

$$n = 2 : \; T(\mathbf{x}) = f(\mathbf{a}) + (f_x(\mathbf{a}), f_y(\mathbf{a})) \cdot (\mathbf{x} - \mathbf{a})$$

In general,

$$T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\nabla f(\mathbf{a}) = (f_1)(\mathbf{a}), f_2(\mathbf{a}, ..., f_n(\mathbf{a})$

Tangent Hyperplane

$n = 1$	Ordinary Tangent Line
$n = 2$	Tangent Plane

Example: $f(x, y, z) = \frac{x^2 y}{z}$
Note: $f : \mathcal{R}^3 \to \mathcal{R}^1$ so GRAPH lives in $\mathcal{R}^4$.
Find Equation of Tangent Hyperplane at $\mathbf{a} = (-3, 4, 2)$

$$f_x(x, y, z) = \frac{2xy}{z}$$

$$f_y(x, y, z) = \frac{x^2}{z} \; so \nabla f(x, y, z) = \left(\frac{2xy}{z}, \frac{x^2}{z}, -\frac{x^y}{z^2} \right)$$

$$f_z(x, y, z) = -\frac{x^y}{z^2}$$

at $\mathbf{a} = (-3, 4, 2)$: $f(\mathbf{a}) = \frac{(-3)^2 \times 4}{2} = 18$

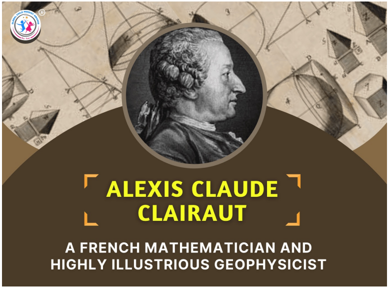
$$\nabla f(\mathbf{a}) = \left(\frac{(2)(-3)(4)}{2}, \frac{(-3)^2}{2}, \frac{-(-3)^2(4)}{2} \right) = \left(-12, \frac{9}{2}, -9 \right)$$

Equation of Tangent Hyperplane is

$$w = 18 + \left(-12, \frac{9}{2}, -9 \right) \cdot (x + 3, y - 4, z - 2)$$

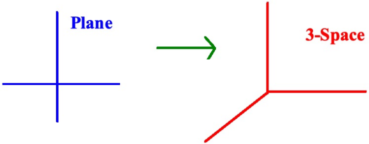
Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$



May 7, 1713 – May 17, 1765

Parametrized Surfaces



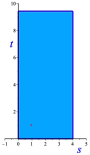
Function from $\mathcal{R}^2 \to \mathcal{R}^3$

Domain:	Patch in Plane
Image:	Surface in Space
Graph:	Lives in $\mathcal{R}^5$

Need for Parametrizations: Graph of $f : \mathcal{R}^1 \to \mathcal{R}^1$ is a curve but not every curve is the graph of such a function

Similarly, graph of $f : \mathcal{R}^2 \to \mathcal{R}^1$ is a surface but not every surface is the graph of such a function.

Example: $\sigma(s,t) = (s\cos t, s\sin t, t), 0 \le s \le 4, 0 \le t \le 3\pi$

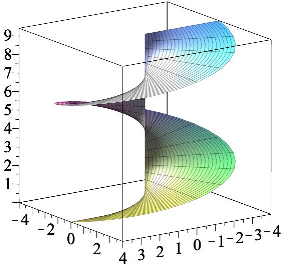


Point: $(1, \pi/4)$ so $\sigma(1, \pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

$\sigma_s(s,t) = (\cos t, \sin t, 0)$ and $\sigma_t(s,t) = (-s\sin t, s\cos t, 1)$

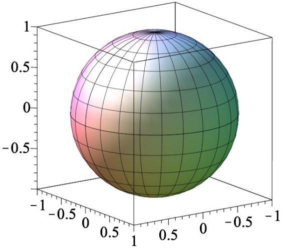
At $\left(1, \frac{\pi}{4}\right)$, representation of the tangent plane is

$$\sigma\left(1, \frac{\pi}{4}\right) + \sigma_s\left(1, \frac{\pi}{4}\right) s + \sigma_t\left(1, \frac{\pi}{4}\right) t$$



Parametrize Unit Sphere

$$\sigma(s,t) = (\cos t \cos s, \sin t \cos s, \sin s), 0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi$$



$$x = \cos t \cos s, y = \sin t \cos s, z = \sin s$$

$$\begin{aligned} x^2 + y^2 + z^2 &= \cos^2 t \cos^2 s + \sin^2 t \cos^2 s + \sin^2 s \\ &= \cos^2 s(\cos^2 t + \sin^2 t) + \sin^2 s \\ &= \cos^2 s + \sin^2 s = 1 \end{aligned}$$

Parametrize Cylinder

$$x = s, y = 4\cos t, z = 4\sin t, 0 \leq s \leq 3, 0 \leq t \leq 2\pi$$