MATH 223: Notes on Sample Exam 2

1. (a) $f:\mathbb{R}^2 \to \mathbb{R}^3$ with $f(x,y) = (x^2y + 2y, xy + 1, \frac{\ln x}{y})$ so f' is the 3×2 matrix

$$\begin{pmatrix} \frac{\partial}{\partial x}(x^2y+2y) & \frac{\partial}{\partial y}(x^2y+2y) \\ \frac{\partial}{\partial x}(xy+1) & \frac{\partial}{\partial y}(xy+1) \\ \frac{\partial}{\partial x}\left(\frac{\ln x}{y}\right) & \frac{\partial}{\partial y}\left(\frac{\ln x}{y}\right) \end{pmatrix} = \begin{pmatrix} 2xy & x^2+2 \\ y & x \\ \frac{1}{xy} & \frac{-\ln x}{y^2} \end{pmatrix} \text{ so } f'(1,2) = \begin{pmatrix} 4 & 3 \\ 2 & 1 \\ 1/2 & 0 \end{pmatrix}$$

(b) $(f \circ g)'(\mathbf{w}) = f'(g(\mathbf{w})g'(\mathbf{w}) = f'(\boldsymbol{x_0})g'(\mathbf{w}) = \begin{pmatrix} 4 & 3 \\ 2 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} -2 & 6 & 4 \\ 2 & -1 & 4 \end{pmatrix} =$

$$\begin{pmatrix} -2 & 21 & 28 \\ -2 & 11 & 12 \\ -1 & 3 & 2 \end{pmatrix}$$

2. (a) Differentiate equations $\begin{matrix} 3x^4y + y^3x^2 - 4t^3 = 4 \\ x + y + 2t - 6 = 0 \end{matrix}$ with respect to t, using the Product Rule and the Chain Rule:

$$\begin{matrix} 12x^3y\frac{dx}{dt} + 3x^4\frac{dy}{dt} + y^3 2x\frac{dx}{dt} + x^2 3y^2\frac{dy}{dt} - 12t^2 = 0 \\ \frac{dx}{dt} + \frac{dy}{dt} + 2 = 0 \end{matrix} \text{ implies}$$

$$\begin{matrix} (12x^3y + 2xy^3)\frac{dx}{dt} + (3x^4 + 3x^2y^2)\frac{dy}{dt} = 12t^2 \\ \frac{dx}{dt} + \frac{dy}{dt} = -2 \end{matrix} \text{ which we evaluate at } t=2,\ x=-1,\ y=3$$

to obtain $\begin{matrix} (-36-54)\frac{dx}{dt} + (3+27)\frac{dy}{dt} = 48 \\ \frac{dx}{dt} + \frac{dy}{dt} = -2 \end{matrix}$ so $\begin{matrix} (-90)\frac{dx}{dt} + (30)\frac{dy}{dt} = 48 \\ \frac{dx}{dt} + \frac{dy}{dt} = -2 \end{matrix}$ which has solution $\frac{dx}{dt} = \frac{-9}{10}$, $\frac{dy}{dt} = \frac{-11}{10}$

(b) An equation for the tangent line is $T(s) = f(2) + f'(2)s = (-1,3) + (-9/10,$-11/10)$s$ = (-1 -9s/10, 3 -11s/10)

3. Let $G(x,y,z) = f(x,y,z) - \lambda(x^2 + y^2 + z^2 - 3^2) = (xz - y^2 + 3x + 3) - \lambda(x^2 + y^2 + z^2 - 3^2)$. Then

$G_x = 0 \Rightarrow z + 3 - 2x\,\lambda = 0 \ \Rightarrow z + 3 = 2x\,\lambda$ (1)

$G_y = 0 \Rightarrow -2y - 2y\,\lambda = 0 \ \Rightarrow -2y = 2y\,\lambda$ (2)

$G_z = 0 \Rightarrow x - 2z\,\lambda = 0 \ \Rightarrow x = 2z\,\lambda$ (3)

$G_\lambda = 0 \Rightarrow x^2 + y^2 + z^2 - 3^2 = 0 \ \Rightarrow x^2 + y^2 + z^2 = 3^2$ (4)

Case 1: If $y \neq 0$, then (2) gives λ = -1 so z+3 = -2x and x=-2z implying z+3 = 4z so z = 1 and then x = -2z = -2. Then using (4): 4 + y^2+1 = 9 so y = 2 or -2. We have two points (x, y, z)= (-2, 2, 1) or (-2, -2, 1) at both of which f has value (-2)(1)-4-6+3 = -9.

Case 2: If y = 0, then we have z+3 = 2 λ x, x=2z λ, and $x^2 + z^2 = 9$.
Hence multiplying the first equation by z gives $z^2 + 3z = 2z\lambda x = xx = x^2 = 9 - z^2$ and we have $2z^2 + 3z - 9 = 0$ or $(2z-3)(z+3) = 0$. Thus $z = \frac{3}{2}$ or $z = -3$
If z = -3, then x = 0 so the point is (0, 0, -3) where f has value 0-0+0+3 = +3 which is not a minimum (from Case 1).
If $z = \frac{3}{2}$, then $x^2 = 9 - z^2 = 9 - \frac{9}{4} = 9 \times \frac{3}{4}$ so $x = \pm\frac{3}{2}\sqrt{3}$. The critical points are $(\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$ and $(-\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$.
At $(\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$, f has the value $\frac{9}{4}\sqrt{3} - 0 + \frac{9}{2}\sqrt{3} + 3 > 3$ so no minimum there either.
At $(-\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$, f has the value $-\frac{9}{4}\sqrt{3} - 0 - \frac{9}{2}\sqrt{3} + 3 = -\frac{27}{4}\sqrt{3} + 3$

Thus the minimum value of f is -9 and it occurs at (-2, 2, 1) and (-2, -2, 1) so you can place the probe at either point.

4. $f(x,y) = 2x^3 + 18y^3 - 18x^2 - 108y^2 + 30x + 162y + 100$.
(a) $f_x(x,y) = 6x^2 - 36x + 30 = 6(x^2 - 6x + 5) = 6(x-5)(x-1)$
$f_y(x,y) = 54y^2 - 216y + 162 = 54(y^2 - 4y + 3) = 54(y-3)(y-1)$
There are 4 critical points where the gradient of f is 0: A:(1,1)   B:(1,3) C:(5,1)   D:(5,3)
(b) $g = \frac{\partial f}{\partial \mathbf{u}} = \nabla f \cdot \mathbf{u} = 6u(x^2 - 6x + 5) + 54v(y^2 - 4y + 3)$ if $\mathbf{u} = (u, v)$. Then $\nabla g = (6u(2x-6),\ 54v(2y-4))$ so $\frac{\partial^2 f}{\partial \mathbf{u}^2} = \nabla g \cdot \mathbf{u} = 6u^2(2x-6) + 54\,v^2\ (2y-4)$ which equals $12u^2(x-3) + 108v^2(y-2)$
(c) At A: $\frac{\partial^2 f}{\partial \mathbf{u}^2} = 12u^2(1-3) + 108v^2(1-2) = -24u^2 - 108v^2 < 0$ for all u,v so f has a relative maximum at (1,1)
At B: $\frac{\partial^2 f}{\partial \mathbf{u}^2} = 12u^2(1-3) + 108v^2(3-2) = -24u^2 + 108v^2 < 0$ which is negative at (u,v) = (1,0) and positive at (u,v) = (0,1) so f has a saddle point at (1,3)
At C: $\frac{\partial^2 f}{\partial \mathbf{u}^2} = 12u^2(5-3) + 108v^2(1-2) = 24u^2 - 108v^2$ which is negative at (u,v) = (0,1) and positive at (u,v) = (1,0) so f has a saddle point at (5,1)
At D: $\frac{\partial^2 f}{\partial \mathbf{u}^2} = 12u^2(5-3) + 108v^2(3-2) = 24u^2 + 108v^2 > 0$ for all u,v so f has a relative minimum at (5,3)

5. (a) $x = r\sin\varphi\cos\theta, y = r\sin\varphi\sin\theta, z = r\cos\varphi$ yields $x^2 + y^2 - z^2 = r^2(sin^2\varphi\, cos^2\theta) + r^2(sin^2\varphi\, sin^2\theta) - r^2(cos^2\varphi) = r^2\, sin^2\varphi\,(cos^2\theta + sin^2\theta) - r^2(cos^2\varphi) = r^2(sin^2\varphi - cos^2\varphi) = -r^2\cos 2\varphi$ so equation is $-r^2\cos 2\varphi = 1$
(b) $r^2 - z^2 = 1$

6. Since $g(t) = (3t^2 + t + 1, 2t, t^2)$, we have $g'(t) = (6t + 1, 2, 2t)$ so $g'(0) = (1, 2, 0)$
This vector has length $\sqrt{5}$ so the unit vector **u** in this direction is $\frac{1}{\sqrt{5}}(1,2,0)$.
The function $f(x,y,z) = x^2 + ye^z$ has gradient $\nabla f(x,y,z) = (2x, e^z, ye^z)$ which has value (2, 1,0) at the point (1,0,0). Thus the directional derivative in the direction **u** is $(2,1,0) \cdot \frac{1}{\sqrt{5}}(1,2,0) = \frac{4}{\sqrt{5}}$.