MATH 223: Multivariable Calculus

Some Notes on Class 1

February 10, 2025

CALCULUS: Limits, Derivatives, Integrals of Functions

Classic Setting: $y = f(x)$ where $f : \mathcal{R}^1 \to \mathcal{R}^1$
Input and Output Are Each Single Numbers
Graph is a CURVE (1 Dimensional) in Plane (R^2)

Idea of a function Generalizes Easily:
Input: One Object
Output: One Object

VECTOR is unifying concept of Linear Algebra and Multivariable Calculus

Calculus I and II: Real-Valued Function of Real Variable
Multivariable Calculus: Vector-Valued Function of Vector Variable

Ultimate Goal: $f : \mathcal{R}^n \to \mathcal{R}^m$.

Example: Let A be a 3×4 matrix and $\mathbf{x}$ a 4×1 vector.

Consider the function $f(\mathbf{x}) = A\mathbf{x}$

$$\underbrace{\begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}}_{\substack{3 \times 4 \\ A}} \underbrace{\begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}}_{\substack{4 \times 1 \\ \mathbf{x}}} = \underbrace{\begin{pmatrix} - \\ - \\ - \end{pmatrix}}_{\substack{3 \times 1 \\ A\mathbf{x}}}$$

Classic Applications of Calculus

- Motion of Object Along Straight Line
  (Position, Velocity, Acceleration)
- Profit as a Function of Price
- Amount of Drug in Bloodstream at time t

Real World Is Much More Complicated

- Motion: Objects Move in Plane or Space
  Need Vector to Describe Location
- Profit: Depends on Prices, Demand, Taxes, Production Costs
- GPA: Function of Many Course Grades

INPUT: Covid Budget
OUTPUT: Amount for Masks, Vaccine, Hospital Equipment, etc.

Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ IF THE LIMIT EXISTS}$$

Example: Find $f'(x)$ if $f(x) = x^3$
Solution:

$$\begin{aligned} f(x+h) - f(x) &= (x+h)^3 - x^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= 3x^2h + 3xh^2 + h^3 \\ &= h\left[3x^2 + 3xh + h^2\right] \end{aligned}$$

$$\text{so } \frac{f(x+h) - f(x)}{h} = \left[3x^2 + 3xh + h^2\right] \text{ if } h \neq 0$$

$$\text{Hence } f'(x) = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Example: Determine $g'(x)$ if $g(x) = x\,f(x)$ where f is a differentiable function.
Solution:

$$\begin{aligned} g(x+h) - g(x) &= (x+h)f(x+h) - x\,f(x) \\ &= x\,f(x+h) + h\,f(x+h) - xf(x) \\ &= x\,[f(x+h) - f(x)] + h\,f(x+h) \end{aligned}$$

Thus The Difference Quotient is

$$\frac{g(x+h) - g(x)}{h} = x\frac{f(x+h) - f(x)}{h} + f(x+h)$$

Taking Limits as $h \to 0$:

$$\frac{f(x+h) - f(x)}{h} \to f'(x)$$

$$f(x+h) \to f(x) \text{ since } f \text{ is continuous}$$

$$\text{Hence } g'(x) = xf'(x) + f(x)$$