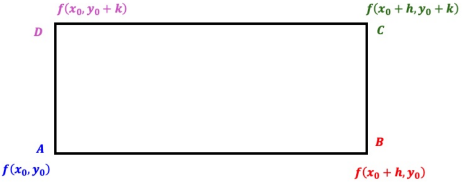
MATH 223: Multivariable Calculus



Class 10: March 3, 2025

- Assignment 10 (For Friday)
- Limits and Continuity

Announcements

Exam 1: Tonight, 7 PM -
No Time Limit

No Books, Notes, Computers, etc.

Warner 100: A - J
Warner 101: K - Z

Today: Begin Chapter 4

Topic: Differentiability

Start with $f : \mathcal{R}^n \to \mathcal{R}^1$

Eventually: $\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$

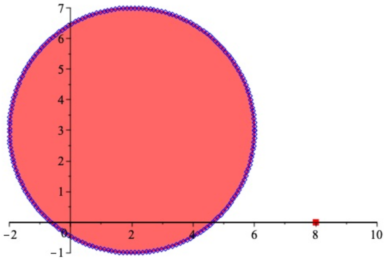
Derivative at point turns out to be $m \times n$ matrix.

But First: Limits and Continuity

Limits and Continuity: Preliminary Concepts

Open Set	Interior Point
Closed Set	Boundary Point
Limit Point	Neighborhood

Example: $S = \{|x - (2,3)| < 4\} \cup \{(8,0)\}$



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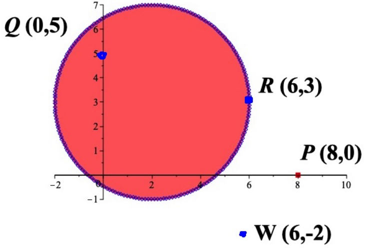

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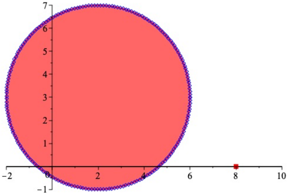


Point	Interior Point?	Limit Point	Boundary Point
Q	Yes	Yes	Yes
R	No	Yes	Yes
P	No	No	Yes
W	No	No	No

Differentiability = Local Linearity = Approximatable By Tangent Object

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ \text{or } f(x) - f(a) &\approx f'(a)(x-a) \\ \text{or } f(x) - f(a) - m(x-a) &\approx 0 \end{aligned}$$

$$\lim_{x \to a} \frac{f(x) - f(a) - m(x-a)}{|x-a|} = 0$$

Generalizing for $\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|} = \mathbf{0}$$

for some $m \times n$ matrix M.

$\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$ is **differentiable** at $\mathbf{a}$ if there exists an $m \times n$ matrix M such that

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}$$

Special Case: $m = 1, n = 2, M$ is 1×2 matrix$\nabla f = (f_x, f_y)$.

Example: $f(x,y) = x^2 + 2xy - y^2$ at (-1,2)

$$f(-1,2) = -7$$

$$f_x(x,y) = 2x + 2y \text{ so } f_x(-1,2) = 2$$

$$f_y(x,y) = 2x - 2y \text{ so } f_y(-1,2) = -6$$

$$\nabla f(-1,2) = (2,-6)$$

Equation of Tangent Plane:

$$\begin{aligned} z &= -7 + (2,-6)\cdot(x+1, y-2) \\ &= -7 + 2x + 2 - 6y + 12 \\ &= +7 + 2x - 6y \end{aligned}$$

Review meaning of $f_x(-1,2) = 2$ and $f_y(-1,2) = 6$

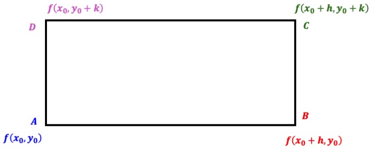
What is rate of change of f at (-1,2) if we approach along direction given by $\mathbf{v} = (3,4)$?

$$
\begin{aligned}
f_{\mathbf{v}}(-1,2) &= \lim_{t\to 0} \frac{f(-1+3t, 2+4t) - f(-1,2)}{t} \\
&= \lim_{t\to 0} \frac{(-1+3t)^2 + 2(-1+3t)(2+4t) - (2+4t)^2 - (-7)}{t} \\
&= \lim_{t\to 0} \frac{17t^2 - 18t}{t} \\
&= \lim_{t\to 0} (17t - 18) = -18
\end{aligned}
$$

Note: $(\nabla f) \cdot \mathbf{v} = (2,-6) \cdot (3,4) = (2)(3) + (-6)(4) = 6 - 24 - 18$

COINCIDENCE?

Clairault's Theorem

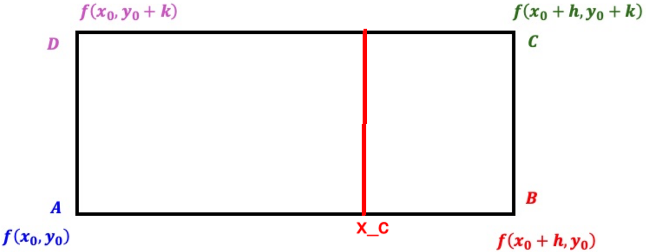


$$H(s,t) = [f(C) - f(B)] - [f(D) - f(A)]$$
$$H(s,t) =$$
$$[f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)] - [f(x_0, y + 0 + k) - f(x_0, y_0)]$$

Define $G(x) = f(x, y_0 + k) - f(x, y_0)$
Apply MVT to G on the interval $[x_0, x_0 + h]$. Result is an x_c inside the interval such $hG'(x_c) = G(x_0 + h) - G(x_0)$

Examine RHS:
$G(x_0 + h) = f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) = f(C) - f(B)$
and $G(x_0) = f(x_0, y_0 + k) - f(x_0, y_0) = f(D) - f(A)$
so $G(x_0 + h) - G(x_0) = F(h, k)$.

LHS: $h\,G'(x_c) = h[f_x(x_c, y_o + k) - f_x(x_c, y_0)]$


$f(x_0, y_0 + k)$
$f(x_0 + h, y_0 + k)$
D
C
A
B
$f(x_0, y_0)$
X_C
$f(x_0 + h, y_0)$

Summarizing,
$hG'(x_c) = G(x_0 + h) - G(x_0)$ becomes
$h[f_x(x_c, y_o + k) - f_x(x_c, y_0)] = F(h, k)$

Now examine the function $H(y) = f_x(x_c, y)$ on the y interval $[y_0, y_0 + k]$
Applying the MVT yields a number y_d inside that interval such that
$k\, H'(y_d) = H(y_0 + k) - H(y_0)$
$k\, f_{xy}(x_c, y_d) = f_x(x_c, y_0 + k) - f_x(x_c, y_0) = \frac{F(h,k)}{h}$
so

$$F(h, k) = kh\, f_{xy}(x_c, y_d)$$

for some point (x_c, y_d) inside rectangle .

For the second half, begin with noting
$H(s,t) = [f(C) - f(B)] - [f(D) - f(A)] =$
$[f(C) - f(D)] - [f(A) - f(B)]$

and following then procedure of the first part, but differentiating first with respect to y and second with respect to x, using MVT twice.

Obtain $F(h,k) = hk\, f_{yx}(x_{c^*}, y_{d^*})$
for some point (x_{c^*}, y_{d^*}) inside rectangle.

Thus $f_{xy}(x_c, y_d) = f_{yx}(x_{c^*}, y_{d^*})$
By Continuity of f_{xy} and f_{yx}, both terms have limit
$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$ as (h,k) approaches (0,0)