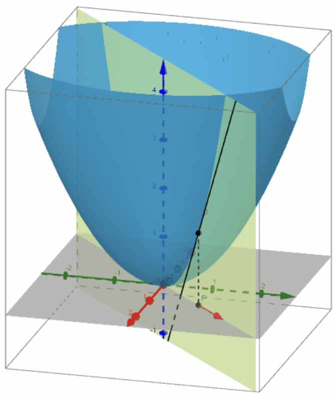
MATH 223: Multivariable Calculus



Class 12: March 7, 2025

Daylight Saving Time Starts
March 9, 2025
Remember to set your clocks ahead one hour Saturday night or Sunday morning the weekend of March 9.

- Notes on Assignment 10
- Assignment 11

Today

Partial With Respect to a Vector

Directional Derivative

Definition: A real-valued function f is differentiable at a point $\mathbf{a}$ means there is a 1 by n matrix $\mathbf{m}$ such that

$$\lim_{|\mathbf{h}| \to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - m\mathbf{h}}{|\mathbf{h}|} = 0$$

Theorem 4.2.1: If f is a real-valued function differentiable at a

point, then $\mathbf{m}$ is the vector of first order partial derivatives evaluated at that point.

Theorem 4..2.2: If f has continuous first order partial derivatives at a point, then f is differentiable at that point.

Partial With Respect to a Vector

Let $f(x,y) = x^2 y$ and $\mathbf{a} = (3,9)$ so $f(3,9) = 81$.
Find the partial derivative of f at (3,9) if we approach (3,9) along arbitrary vector $\mathbf{v} = (v_1, v_2)$.

$$\text{We want } f_{\mathbf{v}}(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}$$

$$\begin{aligned}
f_{\mathbf{v}}(\mathbf{a}) &= \lim_{t \to 0} \frac{f(3 + tv_1, 9 + tv_2) - f(3,9)}{t} \\
&= \lim_{t \to 0} \frac{(3 + tv_1)^2 (9 + tv_2) - (3^2)(9)}{t} \\
&= \lim_{t \to 0} \frac{(3^2 + 6tv_1 + t^2 v_1^2)(9 + tv_2) - (3^2 099)}{t} \\
&= \lim_{t \to 0} \frac{(3^2)(9) + 3^2 tv_2 + 6tv_1(9) + 6t^2 v_1 v_2 + t^2 v_1^2 (9) + t^3 v_1^2 v_2 - (3}{t} \\
&= \lim_{t \to 0} \left(3^2 v_2 + 54 v_1 + 6\mathbf{t} v_1 v_2 + \mathbf{t} v_1^2 (9) + \mathbf{t}^2 v_1^2 v_2\right) \\
&= 9v_2 + 54v_1 = 54v_1 + 9v_2
\end{aligned}$$

Let $f(x, y) = x^2 y$ and $\mathbf{a} = (3, 9)$ so $f(3, 9) = 81$.
Find the partial derivative of f at (3,9) if we approach (3,9) along arbitrary vector $\mathbf{v} = (v_1, v_2)$.

$$f_{\mathbf{v}}(\mathbf{a}) = 54v_1 + 9v_2 = (54, 9) \cdot (v_1, v_2)$$

Note

$$f_x(x, y) = 2xy \text{ and } f_y(x, y) = x^2 \text{ so } f_x(3, 9) = 54, f_y(3, 9) = 9$$

Thus

$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

(In This Case At Least)

Theorem: If $f : \mathcal{R}^n \to \mathcal{R}^1$ is differentiable at $\mathbf{a}$, then

$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Proof of Theorem:
(Case 1): $\mathbf{v} = \mathbf{0}$: Both sides are 0.
(Case 2): $\mathbf{v} \neq \mathbf{0}$:
Note: $|\mathbf{v}| \neq 0$ so we can divide by $|\mathbf{v}|$ if necessary.
By differentiability of f at $\mathbf{a}$, we have

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = 0$$

Set $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ so $\mathbf{x} \to \mathbf{a}$ is equivalent to $t \to 0$ and $\mathbf{x} - \mathbf{a} = t\mathbf{v}$
We have

$$\lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot t\mathbf{v}}{|t\mathbf{v}|} = 0$$

$$\lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot t\mathbf{v}}{|t\mathbf{v}|} = 0$$

Now $|t\mathbf{v}| = |t||\mathbf{v}|$
Can take $t > 0$ (Why?). So $|t\mathbf{v}| = t|\mathbf{v}|$
We can write limit as

$$\lim_{t \to 0} \left[\frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t|\mathbf{v}|} - \frac{t\nabla f(\mathbf{a}) \cdot \mathbf{v}}{t|\mathbf{v}|} \right] = 0$$

Factor out t from second term and multiply both sides by the nonzero scalar $|\mathbf{v}|$ t to obtain

$$\lim_{t \to 0} \left[\frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t} - \nabla f(\mathbf{a}) \cdot \mathbf{v} \right] = 0$$

$$\lim_{t\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a})}{t}-\nabla f(\mathbf{a})\cdot\mathbf{v}\right]=0$$

implies

$$\lim_{t\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a})}{t}\right]=\nabla f(\mathbf{a})\cdot\mathbf{v}$$

But the left hand side is, by definition $f_{\mathbf{v}}(\mathbf{a})$

Directional Derivative

Let $f(x, y) = x^2 y$ and $\mathbf{a} = (3, 9)$ so $f(3, 9) = 81$.
Find the directional derivative of f at (3,9) in the direction of the vector $\mathbf{v} = (v_1, v_2)$.

Recall $f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$

But $\mathbf{w} = 2\mathbf{v}$ points in the same direction as $\mathbf{v}$.
However

$$f_{\mathbf{w}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{w} = 2\nabla f(\mathbf{a}) \cdot \mathbf{v} = 2f_{\mathbf{v}}(\mathbf{a})$$

We want a rate of change that depends only on **DIRECTION**
Idea: Choose a **unit vector u** in that direction that has length 1; that is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Directional Derivative

Let $f(x,y) = x^2y$ and $\mathbf{a} = (3,9)$

Find the directional derivative of f at (3,9) in the direction of the vector $\mathbf{v} = (v_1, v_2)$.

The Directional Derivative is $\nabla f(\mathbf{a}) \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$ where $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$

Normalizing a Vector:

Example $\mathbf{v} = (3,5)$. The $|\mathbf{v}| = \sqrt{3^2 + 5^2} = \sqrt{34}$. The unit vector is

$$\mathbf{u} = \left(\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right)$$

Rate of Change in Direction **u** is

$$\nabla f(\mathbf{a}) \cdot \mathbf{u} = |\nabla f(\mathbf{a})||\mathbf{u}| \cos\theta = |\nabla f(\mathbf{a})| \cos\theta$$

since $|\mathbf{u}| = 1$.

Maximum rate of change occurs when $\cos\theta = 1$; that is $\theta = 0$ so pick **u** in the direction of the gradient.