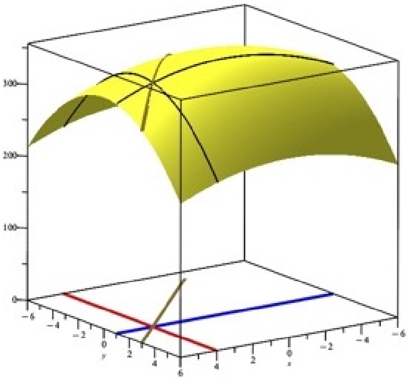
MATH 223: Multivariable Calculus



Class 13: March 10, 2025

Handouts

- Notes on Assignment 11
- Assignment 12

Partial With Respect to a Vector

$f : \mathcal{R}^n \to \mathcal{R}^1$

$\mathbf{a}$ point and $\mathbf{v}$ vector in $\mathcal{R}^n$

Partial derivative $f_{\mathbf{v}}(\mathbf{a})$ of f at $\mathbf{a}$ if we approach $\mathbf{a}$ along vector $\mathbf{v}$

$$\text{We want } f_{\mathbf{v}}(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}$$

Theorem: If f is differentiable at $\mathbf{a}$, then

$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Theorem: If $f : \mathcal{R}^n \to \mathcal{R}^1$ is differentiable at $\mathbf{a}$, then
$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Proof of Theorem:
(Case 1): $\mathbf{v} = \mathbf{0}$: Both sides are 0.
(Case 2): $\mathbf{v} \neq \mathbf{0}$:
Note: $|\mathbf{v}| \neq 0$ so we can divide by $|\mathbf{v}|$ if necessary.
By differentiability of f at $\mathbf{a}$, we have

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = 0$$

Set $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ so $\mathbf{x} \to \mathbf{a}$ is equivalent to $t \to 0$ and $\mathbf{x} - \mathbf{a} = t\mathbf{v}$
We have
$$\lim_{t\to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot t\mathbf{v}}{|t\mathbf{v}|} = 0$$

$$\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot t\mathbf{v}}{|t\mathbf{v}|} = 0$$

Now $|t\mathbf{v}| = |t||\mathbf{v}|$
Can take $t > 0$ (Why?). So $|t\mathbf{v}| = t|\mathbf{v}|$
We can write limit as

$$\lim_{t\to 0} \left[\frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t|\mathbf{v}|} - \frac{t\nabla f(\mathbf{a})\cdot \mathbf{v}}{t|\mathbf{v}|} \right] = 0$$

Factor out t from second term and multiply both sides by the nonzero scalar $|\mathbf{v}|$ t to obtain

$$\lim_{t\to 0} \left[\frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t} - \nabla f(\mathbf{a})\cdot \mathbf{v} \right] = 0$$

$$\lim_{t\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a})}{t}-\nabla f(\mathbf{a})\cdot\mathbf{v}\right]=0$$

implies

$$\lim_{t\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a})}{t}\right]=\nabla f(\mathbf{a})\cdot\mathbf{v}$$

But the left hand side is, by definition $f_{\mathbf{v}}(\mathbf{a})$

Directional Derivative

$$f : \mathcal{R}^n \to \mathcal{R}^1$$

a point and **v** vector in $\mathcal{R}^n$

Find the directional derivative of f at **a** in the direction of the vector **v** is

$$f_{\mathbf{u}}(\mathbf{a}) \text{ where } \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

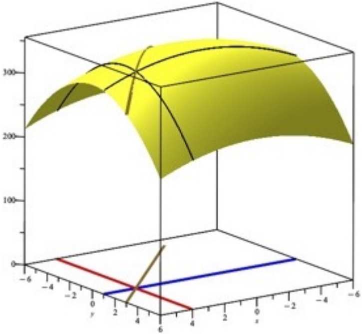
Rate of Change in Direction **u** is

$$\nabla f(\mathbf{a}) \cdot \mathbf{u} = |\nabla f(\mathbf{a})||\mathbf{u}| \cos\theta = |\nabla f(\mathbf{a})| \cos\theta$$

since $|\mathbf{u}| = 1$.

Maximum rate of change occurs when $\cos\theta = 1$; that is $\theta = 0$ so pick **u** in the direction of the gradient.

Maximum Rate of Change Example



$$f(x,y) = 356 - x^2 - 3y^2 \text{ at } \mathbf{a} = (4,1\ )$$
$$f(\mathbf{a}) = f(4,1) = 337$$
$$f_x(x,y) = (-2x), f_y(x,y) = -6y \text{ so } f_x(\mathbf{a}) = -8, f_y(\mathbf{a}) = -6$$
$$\nabla f(\mathbf{a}) = (-8,-6)$$

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v	**u**	Rate of Change
(-1,0)	(-1,0)	8
(0,-1)	(0,-1)	6
(-1,-1)	$(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$	$7\sqrt{2} = 9.89949$
(-3,-2)	$(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}})$	$\frac{36}{\sqrt{13}} = 9.89446$
(-8,-6)	$(\frac{-8}{\sqrt{100}}, \frac{-6}{\sqrt{100}})$	10

Generalized Mean Value Theorem for $f : \mathcal{R}^n \to \mathcal{R}^1$

If f is differentiable at each point of a line segment S between $\mathbf{a}$ and $\mathbf{b}$, then there is a least point $\mathbf{c}$ on S such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Recall classic MVT from Single Variable Calculus:

If $f : \mathcal{R}^1 \to \mathcal{R}^1$ is differentiable on a closed interval $[a, b]$, then there is at least on c inside the interval such that

$$f(b) - f(a) = f'(c)(b - a).$$

An Important Consequence of classic MVT:

Suppose $f'(x) = g'(x)$ for all x in $[a, b]$. Then $f(x) = g(x) + C$ for some constant C and all x in the interval.

Proof: Let $H(x) = f(x) - g(x)$.

Then $H'(x) = f'(x) - g'(x) = 0$ for all x in the interval.

Now let $x_1 < x_2$ be any two points in the interval.

By MVT: $H(_2) - H(x_1) = H'(c)(x_2 - x_1) = 0(x_2 - x_1) = 0$. Thus H is a constant function: $H(x) = C$ for all x.

So $f(x) - g(x) = C$ and hence $f(x) = g(x) + C$.

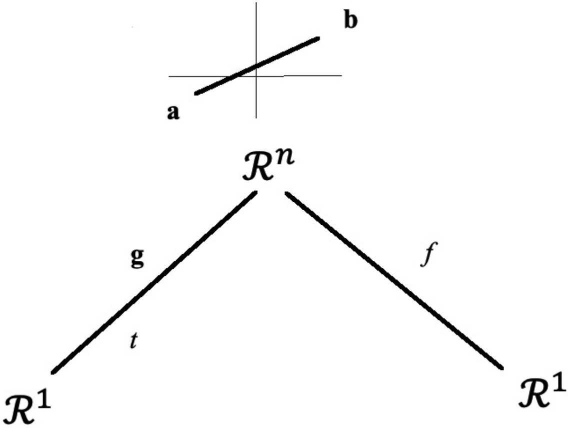
The same argument shows

$$\nabla f \equiv \nabla g \text{ implies } f(\mathbf{x}) = g(\mathbf{x}) + C$$

Generalized Mean Value Theorem for $f : \mathcal{R}^n \to \mathcal{R}^1$

If f is differentiable at each point of a line segment S between **a** and **b**, then there is a least point **c** on S such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$



Proof of Generalized Mean Value Theorem

Define a new function $\mathbf{g} : [0,1] \rightarrow \mathcal{R}^n$ by $\mathbf{g}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$
Note $\mathbf{g}(0) = \mathbf{a}$ and $\mathbf{g}(1) = \mathbf{b}$ and $\mathbf{g}(t)$ lies on S and $\mathbf{g}'(t) = \mathbf{b} - \mathbf{a}$

Consider the composition $H(t) = f(\mathbf{g}(t)) : [0,1] \rightarrow \mathcal{R}^1$
Apply Classic MVT to H:

$$H(1) - H(0) = H'(t_c)(1 - 0) = H'(t_c)$$

but $H(1) = f(g(1)) = f(\mathbf{b})$ and $H(0) = f(g(0)) = f(\mathbf{a})$
Thus $f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c)$
What is $H'(t)$? By Chain Rule: $f'(\mathbf{g}(t))\mathbf{g}'(t) = \nabla f(\mathbf{g}(t)) \cdot (\mathbf{b} - \mathbf{a})$
Let $\mathbf{C} = \mathbf{g}(t_c)$. Then

$$f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c) = \nabla f(\mathbf{C}) \cdot (\mathbf{b} - \mathbf{a})$$