MATH 223: Multivariable Calculus

Class 14: March 12, 2025

- Notes on Assignment 12
- Assignment 13

Today

Generalized Mean Value Theorem

Chain Rule

Implicit Differentiation

Mean Value Theorem for $f : \mathcal{R}^n \to \mathcal{R}^1$

If f is differentiable at each point of a line segment S between $\mathbf{a}$ and $\mathbf{b}$, then there is a least point $\mathbf{c}$ on S such that

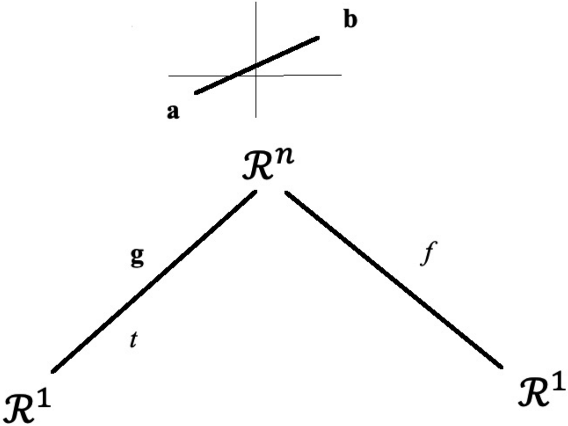
$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Recall classic MVT from Single Variable Calculus:

If $f : \mathcal{R}^1 \to \mathcal{R}^1$ is differentiable on a closed interval $[a, b]$, then there is at least on c inside the interval such that

$$f(b) - f(a) = f'(c)(b - a).$$

Generalized Mean Value Theorem



Proof of Generalized Mean Value Theorem

Define a new function $\mathbf{g} : [0,1] \to \mathcal{R}^n$ by $\mathbf{g}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$
Note $\mathbf{g}(0) = \mathbf{a}$ and $\mathbf{g}(1) = \mathbf{b}$ and $\mathbf{g}(t)$ lies on S and $\mathbf{g}'(t) = \mathbf{b} - \mathbf{a}$

Consider the composition $H(t) = f(\mathbf{g}(t)) : [0,1] \to \mathcal{R}^1$
Apply Classic MVT to H:

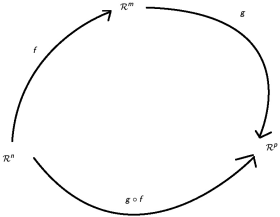
$$H(1) - H(0) = H'(t_c)(1 - 0) = H'(t_c)$$

but $H(1) = f(g(1)) = f(\mathbf{b})$ and $H(0) = f(g(0)) = f(\mathbf{a})$
Thus $f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c)$
What is $H'(t)$? By Chain Rule: $f'(\mathbf{g}(t))\mathbf{g}'(t) = \nabla f(\mathbf{g}(t)) \cdot (\mathbf{b} - \mathbf{a})$
Let $\mathbf{C} = \mathbf{g}(t_c)$. Then

$$f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c) = \nabla f(\mathbf{C}) \cdot (\mathbf{b} - \mathbf{a})$$

An Important Consequence of classic MVT:
Suppose $f'(x) = g'(x)$ for all x in $[a, b]$. Then $f(x) = g(x) + C$
for some constant C and all x in the interval.
Proof: Last Time

The Chain Rule



$$(g \circ f)' = g'(f(x))f'(x)$$

(p x m) (m x n)

matrix matrix

p x n matrix

Example Find $(g \circ f)'$ at $(2,3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ if

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix}, g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ 2u \\ v^2 \end{pmatrix}$$

$$\text{Step I: } f\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2^2 + 6 + 1 \\ 9 + 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$$

$$\text{Step II: } (g \circ f)'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = g'\left(f\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} f'\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$g'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2v \end{pmatrix} \text{ so } g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}$$

$$f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x \\ 0 & 2y \end{pmatrix} \text{ so } f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}$$

$$(g \circ f)'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 14 & 4 \\ 0 & 132 \end{pmatrix}$$

$$(g\circ f)'\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}1&1\\2&0\\0&22\end{pmatrix}\begin{pmatrix}7&2\\0&6\end{pmatrix}=\begin{pmatrix}7&8\\14&4\\0&132\end{pmatrix}$$

Here we can actually check by direct computation:

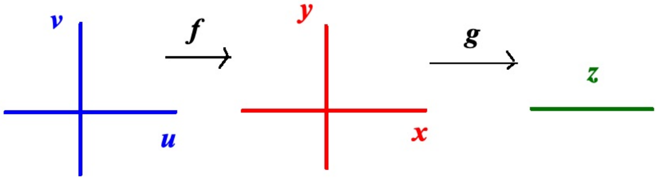
$$g(f(x,y))=g\begin{pmatrix}x^2+xy+1\\y^2+2\end{pmatrix}=\begin{pmatrix}x^2+xy+1+y^2+2\\2x^2+2xy+2\\y^4+4y^2+4\end{pmatrix}$$

$$(g\circ f)'\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+y&x+2y\\4x+2y&2x\\0&4y^3+8y\end{pmatrix}$$

$$(g\circ f)'\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}4+3&2+6\\8+6&4\\0&108+24\end{pmatrix}=\begin{pmatrix}7&2+6\\14&4\\0&132\end{pmatrix}$$

Another Example: Suppose $x = u^2 - v^2, y = 2uv$ and $z = g(x, y)$ for some real-valued differentiable function g.

Show $(z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$



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$$\text{Show } (z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$$

$$\text{Let } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u^2 - y^2 \\ 2uv \end{pmatrix} = f \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\text{Then } f' \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}, g' \begin{pmatrix} x \\ y \end{pmatrix} = (g_x, g_y) = (z_x, z_y)$$

$$\text{Now } (g \circ f)' = g'(f)f' = (z_x, z_y) \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} =$$

$$(2uz_x + 2vz_y, -2vz_x + 2uz_y) = (z_u, z_v)$$

Thus

$$\begin{aligned} z_u^2 + z_v^2 &= 4u^2 z_x^2 + 8uvz_x x_y + 4v^2 z_y^2 + 4v^2 z_x^2 - 8uvz_x z_y + 4u^2 u_z^2 \\ &= 4u^2(z_x^2 + z_y^2) + 4v^2(z_x^2 + z_y^2) = 4(u^2 + v^2)(z_x^2 + z_y^2) \end{aligned}$$

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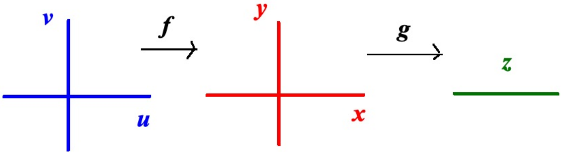


Implicit Differentiation

Example: Find slope of tangent line to the graph of
$4x^2 + 5y^2 = 61$ at (2,3).
(Check point lies on curve: $4(2^2) + 5(3^2) = 16 + 45 = 61$)

A: Direct Solution

$$5y^2 = 61 - 4x^2 \Rightarrow y^2 = \frac{61 - 4x^2}{5} \Rightarrow y = \sqrt{\frac{61 - 4x^2}{5}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{61 - 4x^2}{5}\right)^{-1/2} \frac{-8x}{5}$$

Evaluate at $x = 2$: to get $\frac{1}{2}\left(\frac{45}{5}\right)^{-1/2} \frac{-16}{5} = -\frac{8}{15}$

Implicit Differentiation

Example: Find slope of tangent line to the graph of
$4x^2 + 5y^2 = 61$ at (2,3).

B: Classic Implicit Differentiation

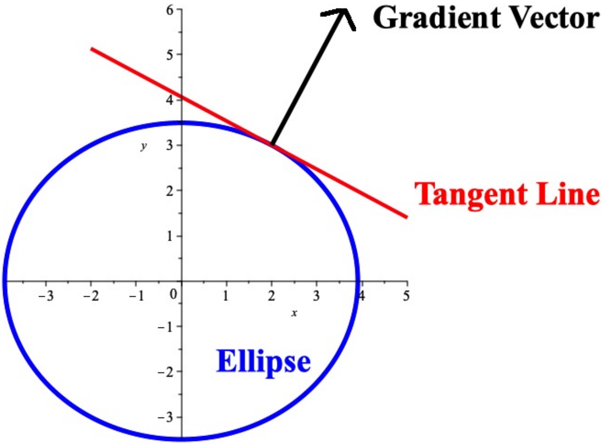
Treat y as an unknown function of x and differentiate:

$$8x + 10y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-8x}{10y} = -\frac{4x}{5y}$$

Evaluate at $x = 2, y = 3$: to get $-\frac{8}{15}$

C: Use Level Curve Idea

If $f(x,y) = 4x^2 + 5y^2$, then (2,3) lies on level curve $f(x,y) = 61$.
Then $\nabla f(2,3)$ is normal to the curve so slope of tangent line is the negative of the slope of the gradient.
$\nabla f(x,y) = (8x, 10y)$ has slope $\frac{10y}{8x} = \frac{15}{8}$ at (2,3). Hence slope of tangent line is $-\frac{8}{15}$.



Gradient Vector
Tangent Line
Ellipse
x
y