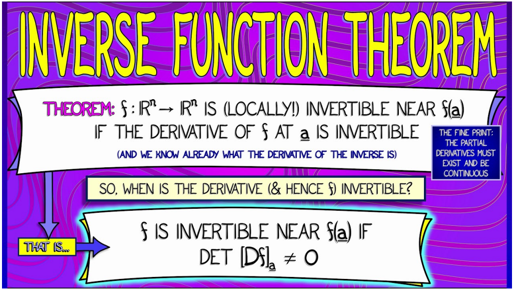
MATH 223: Multivariable Calculus



Class 15: March 14, 2025


HAPPY PI DAY!

- Notes on Assignment 13
- Assignment 14

Review Chain Rule

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Implicit Differentiation

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Change of Variable

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Inverse Function Theorem

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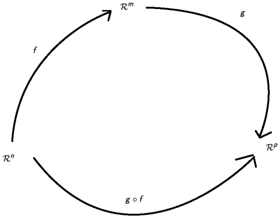
Inverse Function Theorem

Gradient Fields

The Chain Rule



$$(g \circ f)' = g'(f(x))f'(x)$$

(p x m) (m x n)
matrix matrix

p x n matrix

Another Example: Suppose $x = u^2 - v^2, y = 2uv$ and $z = g(x, y)$ for some real-valued differentiable function g.

Show $(z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$

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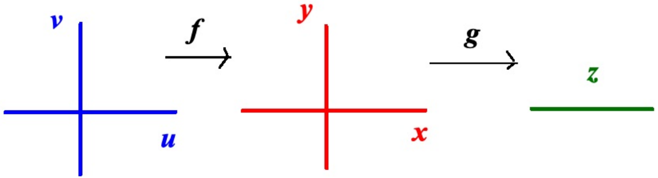
Begin with a picture



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Thus

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Example: Find slope of tangent line to the graph of
$4x^2 + 5y^2 = 61$ at (2,3).
(Check point lies on curve: $4(2^2) + 5(3^2) = 16 + 45 = 61$)

A: Direct Solution

$$5y^2 = 61 - 4x^2 \Rightarrow y^2 = \frac{61 - 4x^2}{5} \Rightarrow y = \sqrt{\frac{61 - 4x^2}{5}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{61 - 4x^2}{5}\right)^{-1/2} \frac{-8x}{5}$$

Evaluate at $x = 2$: to get $\frac{1}{2}\left(\frac{45}{5}\right)^{-1/2} \frac{-16}{5} = -\frac{8}{15}$

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If $f(x, y) = 4x^2 + 5y^2$, then (2,3) lies on level curve $f(x, y) = 61$.

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Then $\nabla f(2, 3)$ is normal to the curve so slope of tangent line is the negative of the slope of the gradient.

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$\nabla f(x, y) = (8x, 10y)$ has slope $\frac{10y}{8x} = \frac{15}{8}$ at (2,3). Hence slope of tangent line is $-\frac{8}{15}$.

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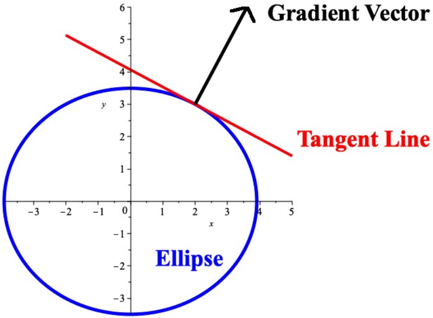


The ellipse is the level curve $F(x, y) = 61$ or $F(x, y) - 61 =$ where $F(x, y) = 4x^2 + 5y^2$.

A piece of the curve around (2,3) is the graph of some implicit function $y = f(x)$.

We want $f'(2)$.

Define a new function $\mathbf{G} : \mathcal{R}^1 \to \mathcal{R}^2$ by

$$\mathbf{G}(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix} \text{ so } \mathbf{G}'(x) = \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$$

Note that this is the tangent vector.

Then $(F \circ \mathbf{G})(x) = 61$ for all x

Take Derivative Using The Chain Rule:

$$F'(\mathbf{G}(x))\mathbf{G}'(x) = 0. \text{Thus } \nabla F(\mathbf{G}(x) \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} = 0$$

Now $G(2) = 3$ and $F(x, y) = 4x^2 + 5y^2$ implies
$\nabla F(x, y) = (8x, 10y)$.
Hence $\nabla F(G(2)) = (8 \times 2, 10 \times 3) = (16, 30)$.
We have $(16, 30) \begin{pmatrix} 1 \\ f'(2) \end{pmatrix} = 0$ so $16 + 30f'(2) = 0$ and thus
$f'(2) = -16/30 = -8/15$.

Change of Variable

Example: Find $\int (10x + 15)^{1/3} dx$

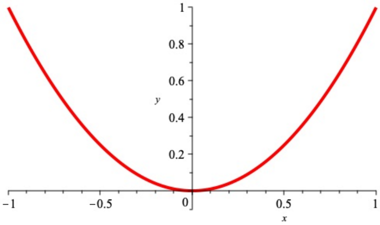
Change of Variable $u = 10x + 15$ so $\mathbf{x = \frac{u-15}{10}}$ and $dx = \frac{1}{10} du$

Integral becomes
$$\int (10x + 15)^{1/3} dx = \int u^{1/3} \frac{1}{10} du = \frac{1}{10} \int u^{1/3} du$$
$$= \frac{1}{10} \times \frac{3}{4} u^{4/3} + C$$
$$= \frac{3}{40} (10x + 15)^{4/3} + C$$

$\mathbf{x = \frac{u-15}{10}}$ is key step. WE MUST BE ABLE TO INVERT THE SUBSTITUTION.

Change of Variable should be invertible, a one-to-one function.

Not Every Function is Invertible

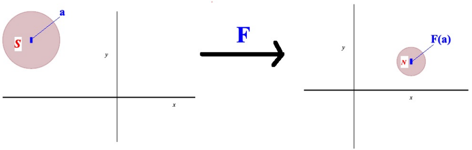


If $y = x^2$, we can not solve unambiguously for x in terms of y globally

$$x = \pm\sqrt{y}$$

but we can solve locally except at origin.

Inverse Function Theorem for $\mathbf{f} : \mathcal{R}^\mathbf{n} \to \mathcal{R}^\mathbf{n}$



IF

- $\mathbf{a}$ is a point in $\mathcal{R}^n$
- S is an open set containing $\mathbf{a}$
- $\mathbf{f}$ is continuously differentiable on S
- Derivative Matrix $\mathbf{f}'(\mathbf{a})$ is invertible

Then

There is a neighborhood N of $\mathbf{a}$ on which $\mathbf{f}^{-\mathbf{1}}$ is defined and

$$\left(\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}))\right)' = [\mathbf{f}'(\mathbf{x})]^{-1} \text{ for all } \mathbf{x} \text{ in } N$$

Example: $\mathbf{f}(x,y) = (\cos x, x\cos x - y)$

$$J = \mathbf{f}'(x,y) = \begin{pmatrix} -\sin x & 0 \\ \cos x - x\sin x & -1 \end{pmatrix}$$

$\det J = \sin x$ so we have invertibility if $x \neq 0, \pi$.

$$(\mathbf{f}^{-1}(x,y))' = J^{-1} = \begin{pmatrix} \frac{-1}{\sin x} & 0 \\ \frac{x\sin x - \cos x}{\sin x} & -1 \end{pmatrix}$$

At $x = \pi/6, y = 2$:

$$f\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\frac{\sqrt{3}}{2} - 2\right) = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}\,\pi}{12} - 2\right)$$

and

$$\mathbf{f}^{-1}(\pi/6, 2))' = \begin{pmatrix} -2 & 0 \\ \frac{\pi}{6} - \sqrt{3} & -1 \end{pmatrix}$$

Gradient Fields

A Gradient Field is just a function from $\mathcal{R}^n$ to $\mathcal{R}^n$ which is the gradient of a differentiable real-valued function.

The gradient $\nabla f(x,y)$ of $f : \mathcal{R}^2 \to \mathcal{R}^2$.

Example 1: $f(x,y) = x^2 \sin y$

Here $\nabla f(x,y) = (2x, x^2 \cos y) = (f_x(x,y), f_y(x,y))$

Note $f_{xy} = x^2 \cos y = f_{yx}$ [Equality of Mixed Partials]

Example 2: Is $\mathbf{F}(x,y) = (y, 2x)$ a gradient field?

If $\mathbf{F} = \nabla f$, then

$$f_x(x,y) = y \implies f_{xy}(x,y) = 1$$
$$f_y(x,y) = 2x \implies f_{yx}(x,y) = 2$$

But these are not equal!

What f we try to build an f by "Partial Integration"?

$$f_x(x,y) = y \implies f(x,y) = xy + G(y) \implies f_y(x,y) = x + G'(y)$$

but we would need G a function of y such that $G'(y) = x$.

We can work backwards on Example 1:
Given $f_x(x, y) = 2x \sin y$, "partial integration" with respect to x produces $f(x, y) = x^2 \sin y + G(y)$ and that yields $f_y = x^2 \cos y + G'(y)$ which equals $x^2 \cos y$ by choosing G to be any constant function.


happy
Spring
Break