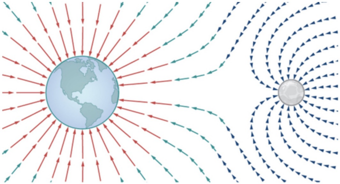
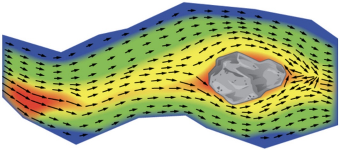
MATH 223: Multivariable Calculus



(a)



(b)

Class 16: March 24, 2025

Welcome Back!

We hope you enjoyed
Spring Break.

- Notes on Assignment 14
- Assignment 15

Exam 2 Next Monday

Change of Variable

Example: Find $\int (10x+15)^{1/3} dx$

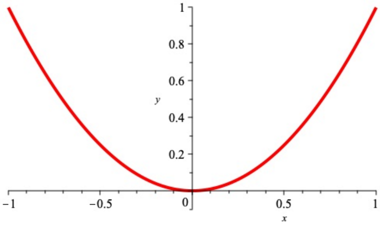
Change of Variable $u = 10x + 15$ so $\mathbf{x} = \frac{\mathbf{u-15}}{\mathbf{10}}$ and $dx = \frac{1}{10} du$

Integral becomes
$$\int (10x+15)^{1/3} dx = \int u^{1/3} \frac{1}{10} du = \frac{1}{10} \int u^{1/3} du$$
$$= \frac{1}{10} \times \frac{3}{4} u^{4/3} + C$$
$$= \frac{3}{40} (10x+15)^{4/3} + C$$

$\mathbf{x} = \frac{\mathbf{u-15}}{\mathbf{10}}$ is key step. WE MUST BE ABLE TO INVERT THE SUBSTITUTION.

Change of Variable should be invertible, a one-to-one function.

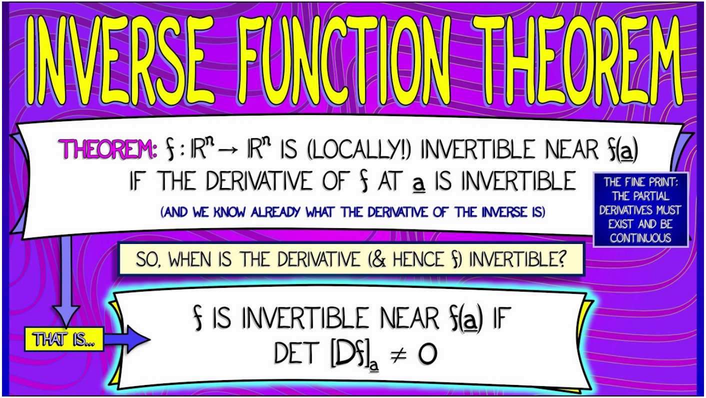
Not Every Function is Invertible



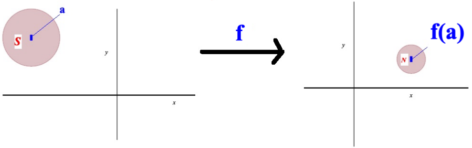
If $y = x^2$, we can not solve unambiguously for x in terms of y globally

$$x = \pm\sqrt{y}$$

but we can solve locally except at origin.


INVERSE FUNCTION THEOREM
THEOREM: f : IRⁿ → IRⁿ IS (LOCALLY!) INVERTIBLE NEAR f(a)
IF THE DERIVATIVE OF f AT a IS INVERTIBLE
(AND WE KNOW ALREADY WHAT THE DERIVATIVE OF THE INVERSE IS)
THE FINE PRINT: THE PARTIAL DERIVATIVES MUST EXIST AND BE CONTINUOUS
SO, WHEN IS THE DERIVATIVE (& HENCE f) INVERTIBLE?
THAT IS...
f IS INVERTIBLE NEAR f(a) IF
DET [Df]a ≠ 0

Inverse Function Theorem for $\mathbf{f} : \mathcal{R}^{\mathbf{n}} \to \mathcal{R}^{\mathbf{n}}$



IF

- $\mathbf{a}$ is a point in $\mathcal{R}^n$
- S is an open set containing $\mathbf{a}$
- $\mathbf{f}$ is continuously differentiable on S
- Derivative Matrix $\mathbf{f}'(\mathbf{a})$ is invertible

Then

There is a neighborhood N of $\mathbf{f}(\mathbf{a})$ on which $\mathbf{f}^{-1}$ is defined and

$$\left(\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}))\right)' = [\mathbf{f}'(\mathbf{x})]^{-1} \text{ for all } \mathbf{x} \text{ in } N$$

Example: $\mathbf{f}(x,y) = (\cos x, x\cos x - y)$

$$J = \mathbf{f}'(x,y) = \begin{pmatrix} -\sin x & 0 \\ \cos x - x\sin x & -1 \end{pmatrix}$$

$\det J = \sin x$ so we have invertibility if $x \neq 0, \pi$.

$$(\mathbf{f}^{-1}(x,y))' = J^{-1} = \begin{pmatrix} \frac{-1}{\sin x} & 0 \\ \frac{x\sin x - \cos x}{\sin x} & -1 \end{pmatrix}$$

At $x = \pi/6, y = 2$:

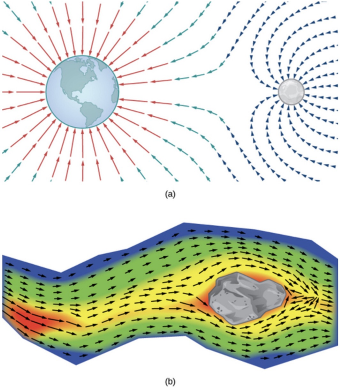
$$f\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\frac{\sqrt{3}}{2} - 2\right) = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}\,\pi}{12} - 2\right)$$

and

$$\mathbf{f}^{-1}(\pi/6, 2))' = \begin{pmatrix} -2 & 0 \\ \frac{\pi}{6} - \sqrt{3} & -1 \end{pmatrix}$$

Vector Fields

A **Vector Field** is just a function F from $\mathcal{R}^n$ to$\mathcal{R}^n$



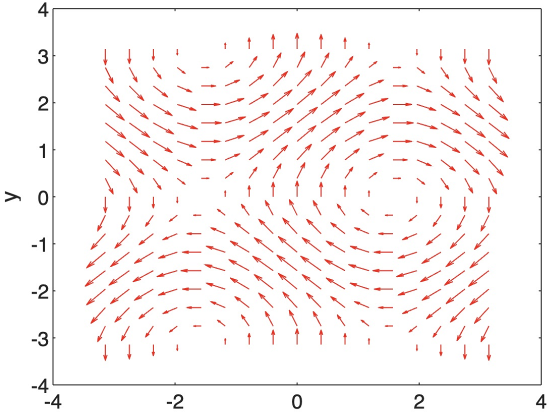
(a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Example: $f : \mathcal{R}^2 \to \mathcal{R}^2$

$$F(x, y) = (siny, cosx)$$

In MATLAB

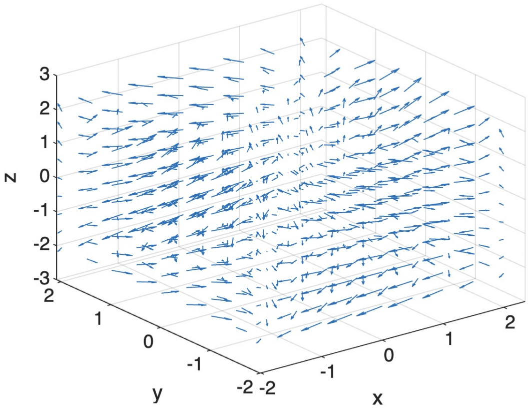
```
[x,y] = meshgrid(-pi:pi/8:pi,-pi:pi/8:pi);
quiver(x,y,sin(y),cos(x),'r')
xlabel('x')
ylabel('y')
```



Example: $f : \mathcal{R}^3 \to \mathcal{R}^3$

$$F(x, y, z) = (x^2 - y^2, \cos 3y, z)$$

```
[x,y,z]  = meshgrid(-2: .5 : 2, -2: .5 : 2, -2 :.5 :2);
quiver3(x,y,z,x.^2 - y.^2, cos(3 * y) ,z)
xlabel('x')
ylabel('y')
zlabel('z')
```



Gradient Fields

A **Vector Field** is just a function F from $\mathcal{R}^n$ to $\mathcal{R}^n$. A **Gradient Field** is a vector field which is the gradient of a real-valued function.

If f is a real-valued function of n variables such that $\nabla f = \mathbf{F}$, then f is a called a **potential** of $\mathbf{F}$.

Gradient Fields

A **Gradient Field** is a vector field which is the gradient of a real-valued function.

The gradient $\nabla f(x, y)$ of $f : \mathcal{R}^2 \to \mathcal{R}^2$.

Example 1: $f(x, y) = x^2 \sin y$

Here $\nabla f(x, y) = (2x, x^2 \cos y) = (f_x(x, y), f_y(x, y))$

Note $f_{xy} = x^2 \cos y = f_{yx}$ [Equality of Mixed Partials]

Example 2: Is $\mathbf{F}(x, y) = (y, 2x)$ a gradient field?

If $\mathbf{F} = \nabla f$, then

$$f_x(x, y) = y \implies f_{xy}(x, y) = 1$$
$$f_y(x, y) = 2x \implies f_{yx}(x, y) = 2$$

But these are not equal!

What f we try to build an f by "Partial Integration"?

$f_x(x, y) = y \implies f(x, y) = xy + G(y) \implies f_y(x, y) = x + G'(y)$

but we would need G a function of y such that $G'(y) = x$.

We can work backwards on Example 1:
Given $f_x(x, y) = 2x \sin y$, "partial integration" with respect to x produces $f(x, y) = x^2 \sin y + G(y)$ and that yields $f_y = x^2 \cos y + G'(y)$ which equals $x^2 \cos y$ by choosing G to be any constant function.

Example: Find a potential function f if

$$\nabla f(x,y) = (2x\ln(xy) + x - y^3, \frac{x^2}{y} - 3y^2x)$$

Step 1: Check Equality Of Mixed Partials

$$f_x(x,y) = 2x\ln(xy) + x - y^3 \implies f_{xy} = 2x\frac{1}{xy}x - 3y^2 = \frac{2x}{y} - 3y^2$$

$$f_y(x,y) = \frac{x^2}{y} - 3y^2x \implies f_{yx} = \frac{2x}{y} - 3y^2$$

Step 2: Integrate with respect to one of the variables
Here we will integrate f_y with respect to y so f has the form

$$f(x,y) = \int \frac{x^2}{y} - 3y^2x\,dy = x^2\ln y - y^3x + H(x)$$

for some function H of x.

Step 3: Take partial derivative of the result of Step 2 with respect to the other variable to see how close we are to the result we want.
Fix the difference by adjusting the "constant" of integration.
With $f(x,y) = x^2\ln y - y^3x + H(x)$, we have

$$f_x(x,y) = 2x\ln y - y^3 + H'(x)$$

With $f(x,y) = x^2 \ln y - y^3 x + H(x)$, we have

$$f_x(x,y) = 2x \ln y - y^3 + H'(x)$$

which we want equal to

$$2x \ln(xy) + x - y^3 = 2x \ln x + 2x \ln y + x - y^3$$

Thus we need $H'(x) = 2x \ln x + x$ so we can take
$H(x) = x^2 \ln x + C$

Step 4: Put it all together to form a potential function:

$$f(x,y) = x^2 \ln y - y^3 x + H(x) = x^2 \ln y - y^3 x + x^2 \ln x + C$$