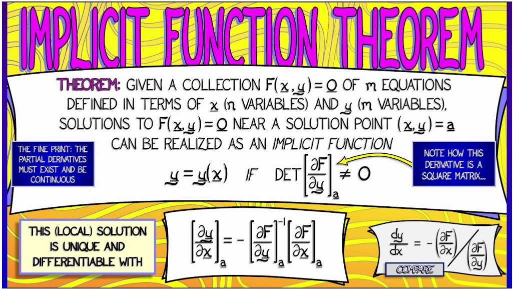
MATH 223: Multivariable Calculus



Class 17: March 26, 2025

- Notes on Assignment 15
- Assignment 16

Exam 2 Next Monday Night

Today

Implicit Differentiation II

Implicit Function Theorem

Implicit Differentiation II

The Surface $2x^3y + yx^2 + t^2 = 0$ and the Plane $x + y + t - 1 = 0$

intersect along a Curve which contains the point
$t = 1, x = -1, y = 1$

Check: Surface: $2(-1)(1) + 1(-1)^2 + 1^2 = 0$; Plane:
$-1 + 1 + 1 - 1 = 0$
Treat x and y as unknown functions of t.
Problem: Find $x'(t)$ and $y'(t)$ at $(t, x, y) = (1, -1, 1)$

Each equation defines a surface in 3-space and intersection of two surfaces is a curve.
The curve has some parametrization **G**

$$\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \mathcal{R}^1 \to \mathcal{R}^3$$

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$$\text{Consider } \mathcal{R}^1 \xrightarrow{\mathbf{G}} \mathcal{R}^3 \xrightarrow{\mathbf{F}} \mathcal{R}^2$$

$$\text{where } \mathbf{F}(x, y, t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix} = \begin{pmatrix} 2x^3y + yx^2 + t^2 \\ x + y + t - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $\mathbf{F}(\mathbf{G}(t)) = \mathbf{0}$ for all t

Differentiate using Chain Rule:

$$[\mathbf{F}(\mathbf{G}(t))]' = \mathbf{F}'(\mathbf{G}(t))\mathbf{G}'(t) = \begin{pmatrix} F_{1t} & F_{1x} & F_{1y} \\ F_{2t} & F_{2x} & F_{yt} \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2t & 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write

$$\begin{pmatrix} 2t & 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as

$$\begin{pmatrix} 2t \\ 1 \end{pmatrix} + \begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2t \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

Multiply each side by inverse of coefficient matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

Evaluate at the given point: $t = 1, x = -1, y = 1$

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= -\begin{pmatrix} 6-2 & -2+1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -2/5 \end{pmatrix} \end{aligned}$$

More Generally

$$\left\{\begin{matrix} F_1(x,y,t)=0 \\ F_2(x,y,t)=0 \end{matrix}\right\} \text{ define } x, y \text{ implicitly as functions of } t$$

<u>Problem</u>: Find $x'(t)$ and $y'(t)$ where $\mathbf{f}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$.

Set Up: $\mathcal{R}^1 \overset{\mathbf{G}}{\rightarrow} \mathcal{R}^3 \overset{\mathbf{F}}{\rightarrow} \mathcal{R}^2$ where $\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \mathbf{F}(t,x,y) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

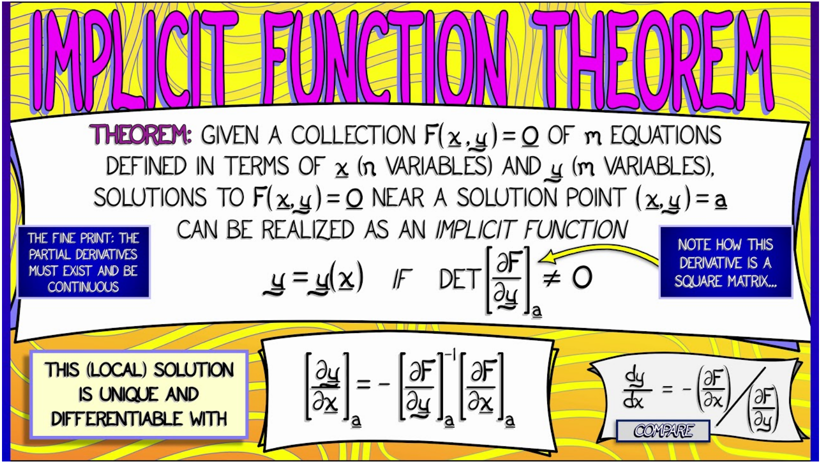
Then $\mathbf{F}(\mathbf{G}(t)) \equiv 0$ so $\mathbf{F}'(\mathbf{G}(t))\mathbf{G}'(t) = 0$ which we write as

$$(F_t, F_x, F_y)\begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = 0 \text{ or } F_t + [F_x, F_y][\mathbf{f}'(t)] = 0$$

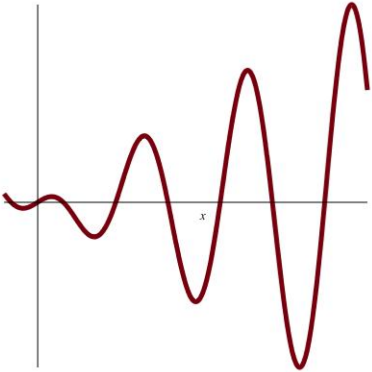
$$\mathbf{f}'(t) = -[F_x, F_y]^{-1} F_t$$

Here the notation is

$$F_x = \begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix}, F_y = \begin{pmatrix} F_{1y} \\ F_{2y} \end{pmatrix}, F_t = \begin{pmatrix} F_{1t} \\ F_{2t} \end{pmatrix}$$


IMPLICIT FUNCTION THEOREM
THEOREM: GIVEN A COLLECTION $F(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{0}$ OF m EQUATIONS DEFINED IN TERMS OF $\underset{\sim}{x}$ (n VARIABLES) AND $\underset{\sim}{y}$ (m VARIABLES), SOLUTIONS TO $F(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{0}$ NEAR A SOLUTION POINT $(\underset{\sim}{x}, \underset{\sim}{y}) = \underline{a}$ CAN BE REALIZED AS AN *IMPLICIT FUNCTION*
$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x})$ IF $\mathrm{DET}\left[\frac{\partial F}{\partial \underset{\sim}{y}}\right]_{\underline{a}} \neq 0$
THE FINE PRINT: THE PARTIAL DERIVATIVES MUST EXIST AND BE CONTINUOUS
NOTE HOW THIS DERIVATIVE IS A SQUARE MATRIX...
THIS (LOCAL) SOLUTION IS UNIQUE AND DIFFERENTIABLE WITH
$\left[\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right]_{\underline{a}} = -\left[\frac{\partial F}{\partial \underset{\sim}{y}}\right]_{\underline{a}}^{-1}\left[\frac{\partial F}{\partial \underset{\sim}{x}}\right]_{\underline{a}}$
$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial x}\right) \Big/ \left(\frac{\partial F}{\partial y}\right)$
COMPARE

Today:
Maxima and Minima of Real-Valued Functions

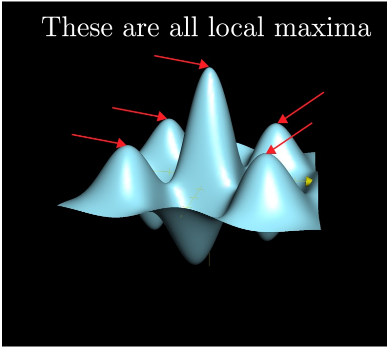


Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^1$ be a real-valued function with $\vec{x_o}$ a point in D.

Definition: f has an **absolute maximum** at $\vec{x_o}$ if $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in D.

Note: $\geq$ makes sense because we are comparing real numbers.

f has a **relative maximum** at $\vec{x_o}$ if there is a neighborhood N around $\vec{x_o}$ such that $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in N.



Theorem: Let $\vec{x_o}$ be an **interior** point of D. If f is differentiable at $\vec{x_o}$ and f has a relative maximum or minimum at $\vec{x_o}$, then $f'(\vec{x_o}) = \nabla(\vec{x_o}) = \vec{0}$.

Proof: Suppose f has a relative maximum at $\vec{x_o}$

Let $\vec{u}$ be any unit vector in $\mathbb{R}^n$.

$$\text{Then } \frac{\partial f}{\partial \vec{u}} = \lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{u}) - f(\vec{x_0})}{t}$$

(a) Take $\lim\limits_{t \to 0^+} : \frac{-}{+} \leq 0$

thus $\frac{\partial f}{\partial \vec{u}} = 0$ for all $\vec{u}$

(b) Take $\lim\limits_{t \to 0^-} : \frac{-}{-} \geq 0$

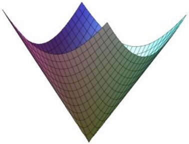
Taking $\vec{u}$ to be unit vectors gives $\nabla f(\vec{x_0}) = 0$

Theorem: f differentiable at relative extrema implies gradient is 0.

The Theorem Has Its Limitations:

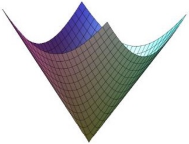
(1) **The function can have an extreme value at a point where it is not differentiable.**

Example: $f(x,y)=\sqrt{x^2+y^2}$ has minimum at (0,0) but is not differentiable there.



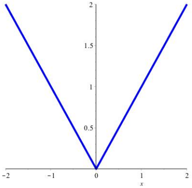
$f_x(x,y)=\frac{x}{\sqrt{x^2+y^2}}, f_y(x,y)=\frac{y}{\sqrt{x^2+y^2}},$

Example: $f(x,y) = \sqrt{x^2 + y^2}$ has minimum at (0,0) but is not differentiable there.



Analogue in Calculus I:

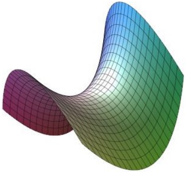
$f(x) = \sqrt{x^2} = |x|$

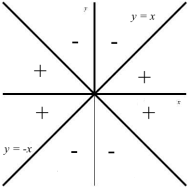


Theorem: f differentiable at relative extrema implies gradient is 0.

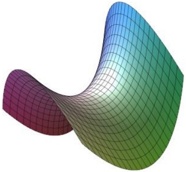
The Theorem Has Its Limitations:

(2) We can have $\nabla f(\vec{x_0}) = 0$ but no extreme point at $\vec{x_0}$





$$\nabla f(x, y) = (2x, 2y)$$



There is a Maximum is one direction and a Minimum in another

Saddle Point

Quiz:
Name a Famous Commercial Food Product That Exhibits a Saddle Point


Pringles
the
Original
NET WT/PESO NETO 2.36 OZ

Definition: A point $\vec{x_0}$ in the domain of f is a **Critical Point** of f if

(a) $\nabla f(\vec{x_0}) = \vec{0}$

or

(b) ∇f does not exist at $\vec{x_0}$.

Extreme Values Can Occur at Critical Points or Points on the Boundary

Example: Temperature Distribution on disk of radius 1 centered at origin is $T(x, y) = 2x^2 + 4y^2 + 2x + 1$.

For Critical Points, examine $\nabla T = (4x + 2, 8y)$

$\nabla T = (0, 0)$ only at $x = -\frac{1}{2}, y = 0$

which does lie inside the disk.

Note $T(-\frac{1}{2}, 0) = 2(\frac{1}{4}) + 4(0^2) + 2(-\frac{1}{2}) + 1 = \frac{1}{2}$, and $T(0, 0) = 1$.

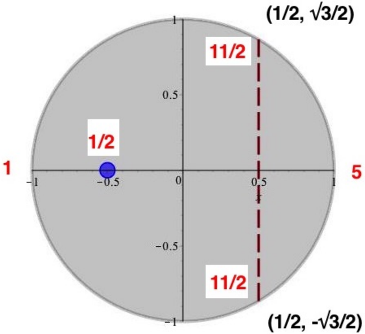
Analyze Along Boundary:

$x^2 + y^2 = 1$ so $y^2 = 1 - x^2$ and

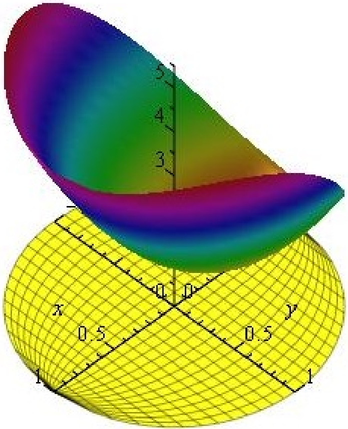
$T(x, y) = g(x) = 2x^2 + 4(1 - x^2) + 2x + 1 = -2x^2 + 2x + 5$

Thus $g'(x) = -4x + 2, g''(x) = -4$ so $x = \frac{1}{2}$ gives a maximum.

$x = \frac{1}{2}$ gives $y^2 = 1 - \frac{1}{4} = \frac{3}{4}$ so $y = \pm\frac{\sqrt{3}}{2}$



red numbers are values of the function


5
4
3
0
0
x
y
0.5
0.5
1
1

Parametrize Boundary

$x = \cos t, y = \sin t$ for $0 \le t \le 2\pi$

$$
\begin{aligned}
T(x,y) &= 2x^2 + 4y^2 + 2x + 1 \\
&= 2\cos^2 t + 4\sin^2 t + 2\cos t + 1 \\
&= 2\cos^2 t + 2\sin^2 t + 2\sin^2 t + 2\cos t + 1 \\
&= 2 + 2\sin^2 t + 2\cos t + 1 = 2\sin^2 t + 2\cos t + 3 \\
&= H(t)
\end{aligned}
$$

$H(0) = 2\cdot 1 + 2\cdot 0 + 3 = 5, H(\pi) = 2\cdot 1 + 2\cdot -1 + 3 = 1$

Now $H'(t) = 4\sin t\cos t - 2\sin t = 2\sin t(2\cos t - 1)$ so

$H'(t) = 0$ at $\sin t = 0$ or $\cos t = \frac{1}{2}$

The first condition gives $t = 0, t = \pi$, the second occurs when $t = \frac{\pi}{3}$.

Next Time:

Solving Constrained Optimization Problems Using Lagrange Multipliers