MATH 223: Multivariable Calculus

Joseph–Louis Lagrange

Class 18: March 28, 2025

Notes on Assignment 16

Assignment 17

Announcements

Exam 2: Monday Evening at 7 PM

Extreme Values

Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^1$ be a real-valued function with $\vec{x_o}$ a point in D.

Definition: f has an **absolute maximum** at $\vec{x_o}$ if $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in D.

Note: $\geq$ makes sense because we are comparing real numbers.

f has a **relative maximum** at $\vec{x_o}$ if there is a neighborhood N around $\vec{x_o}$ such that $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in N.

Theorem: Let $\vec{x_o}$ be an **interior** point of D. If f is differentiable at $\vec{x_o}$ and f has a relative maximum or minimum at $\vec{x_o}$, then $f'(\vec{x_o}) = \nabla(\vec{x_o}) = \vec{0}$.

Theorem: f differentiable at relative extrema implies gradient is 0.

The Theorem Has Its Limitations:

(1) **The function can have an extreme value at a point where it is not differentiable.**

(2) We can have $\nabla f(\vec{x_0}) = 0$ but no extreme point at $\vec{x_0}$

There is a Maximum is one direction and a Minimum in another

Saddle Point

Definition: A point $\vec{x_0}$ in the domain of f is a **Critical Point** of f if

(a) $\nabla f(\vec{x_0}) = \vec{0}$

or

(b) ∇f does not exist at $\vec{x_0}$.

Extreme Values Can Occur at Critical Points or Points on the Boundary

Example: Temperature Distribution on disk of radius 1 centered at origin is $T(x, y) = 2x^2 + 4y^2 + 2x + 1$.

For Critical Points, examine $\nabla T = (4x + 2, 8y)$

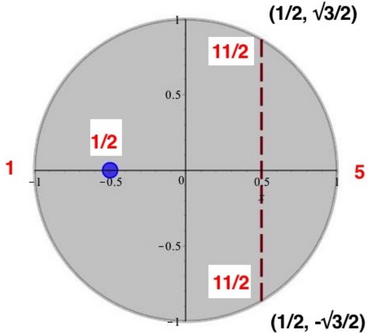
$\nabla T = (0, 0)$ only at $x = -\frac{1}{2}, y = 0$

which does lie inside the disk.

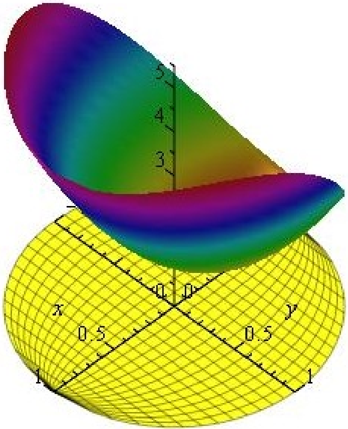
Note $T(-\frac{1}{2}, 0) = 2(\frac{1}{4}) + 4(0^2) + 2(-\frac{1}{2}) + 1 = \frac{1}{2}$, and $T(0, 0) = 1$.

Analyze Along Boundary:

$$x^2 + y^2 = 1 \text{ so } y^2 = 1 - x^2 \text{ and}$$
$$T(x, y) = g(x) = 2x^2 + 4(1 - x^2) + 2x + 1 = -2x^2 + 2x + 5$$
$$\text{Thus } g'(x) = -4x + 2, g''(x) = -4 \text{ so } x = \tfrac{1}{2} \text{ gives a maximum.}$$
$$x = \tfrac{1}{2} \text{ gives } y^2 = 1 - \tfrac{1}{4} = \tfrac{3}{4} \text{ so } y = \pm\tfrac{\sqrt{3}}{2}$$



red numbers are values of the function



5
4
3
0
0
x
y
0.5
0.5
1
1

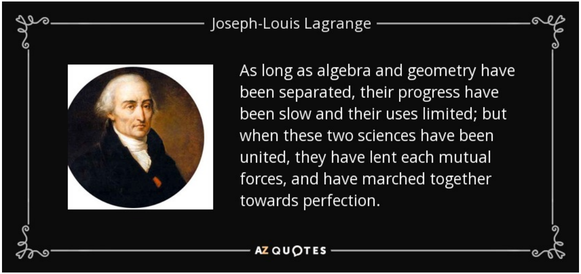
Parametrize Boundary

$x = \cos t, y = \sin t$ for $0 \leq t \leq 2\pi$

$$\begin{aligned} T(x,y) &= 2x^2 + 4y^2 + 2x + 1 \\ &= 2\cos^2 t + 4\sin^2 t + 2\cos t + 1 \\ &= 2\cos^2 t + 2\sin^2 t + 2\sin^2 t + 2\cos t + 1 \\ &= 2 + 2\sin^2 t + 2\cos t + 1 = 2\sin^2 t + 2\cos t + 3 \\ &= H(t) \end{aligned}$$

$H(0) = 2\cdot 1 + 2\cdot 0 + 3 = 5, H(\pi) = 2\cdot 1 + 2\cdot -1 + 3 = 1$
Now $H'(t) = 4\sin t\cos t - 2\sin t = 2\sin t(2\cos t - 1)$ so
$H'(t) = 0$ at $\sin t = 0$ or $\cos t = \frac{1}{2}$
The first condition gives $t = 0, t = \pi$, the second occurs when $t = \frac{\pi}{3}$.

Solving Constrained Optimization Problems Using Lagrange Multipliers

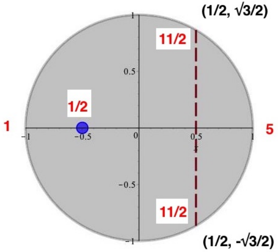


Constrained Optimization:

Via Method of Lagrange Multipliers

$T(x,y) = 2x^2 + 4y^2 + 2x + 1$ on unit disk

$D = \{(x,y) : x^2 + y^2 \leq 1\}$



Revisit Problem

Find Extreme Values of

$T(x, y) = 2x^2 + 4y^2 + 2x + 1$ on unit disk

$D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Findings: Maximum Value of $5\frac{1}{2}$ at $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

Minimum Value of $\frac{1}{2}$ at $(-\frac{1}{2}, 0)$

Lagrange Multiplier Method

Joseph-Louis Lagrange (1736 – 1813)

(Actually Used by Euler 40 years before Lagrange)

Find Extreme Values of
$T(x,y) = 2x^2 + 4y^2 + 2x + 1$ on unit disk
$D = \{(x,y) : x^2 + y^2 \leq 1\}$.

Let $F(x,y,\lambda) = 2x^2 + 4y^2 + 2x + 1 + \lambda(x^2 + y^2 - 1)$

$F_x = 4x + 2 + 2\lambda x$ **(1)** $4x + 2 + 2x\lambda = 0$

$F_y = 8y + 2\lambda y$ $\nabla F = \vec{0}$ implies   **(2)** $8y + 2y\lambda = 0$

$F_\lambda = x^2 + y^2 - 1$ **(3)** $x^2 + y^2 = 1$

Now **(2)** gives $2y(4+\lambda) = 0$ so $y = 0$ or $\lambda = -4$

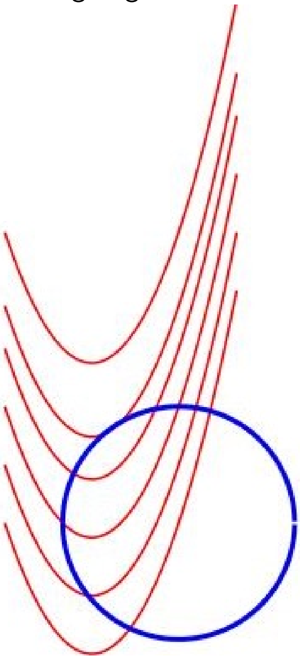
Then $y = 0$ makes **(3)** $x^2 + 0 = 1$ yielding $x = \pm 1$

This gives two points: (1,0) and (-1,0)

Now $\lambda = -4$ gives **(1)** $4x + 2 - 8x = 0 \implies 4x = 2 \implies x = \frac{1}{2}$

and then **(3)** yields $\frac{1}{4} + y^2 = 1 \implies y^2 = \frac{3}{4} \implies y = \pm\frac{\sqrt{3}}{2}$

Lagrange's Idea



Maple Examples

Look at a More General Problem
Maximize f(x,y) [Objective Function]
Subject to g(x,y) = C [Constraint]

Examine Level Curves of f $(f(x,y) = k$

Find intersection of constraint curve with level curve that has largest k.

This appears to occur at a point where the two curves are tangent to each other;

that is, gradient vectors point in the same direction.

$$\text{Hence } f'(\vec{x_0}) = \lambda g'(\vec{x_0}) \text{ for some } \lambda$$

$$\text{so that } f'(\vec{x_0}) - \lambda g'(\vec{x_0}) = \vec{0}$$

Forming the function $F(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - C]$ and look for critical points of F

$\nabla F = \vec{0}$ where:

$$F_x : f_x = \lambda g_x$$
$$F_y : f_y = \lambda g_y$$
$$F_\lambda : g(x, y) = C$$

Example: Find Extreme Values of $f(x, y, z) = x + y + z$

subject to

$x^2 + y^2 = 2$ and

$x + z = 1$

Let $F(x, y, z, \lambda, \mu) = x + y + z + \lambda(x^2 + y^2 - 2) + \mu(x + z - 1)$

$$
\text{Then} \quad
\begin{array}{lll}
F_x = 0 & 1 + 2\lambda x + \mu = 0 & \textbf{(1)} \\
F_y = 0 & 1 + 2\lambda y = 0 & \textbf{(2)} \\
F_z = 0 & 1 + \mu = 0 & \textbf{(3)} \\
F_\lambda = 0 & x^2 + y^2 = 2 & \textbf{(4)} \\
F_\mu = 0 & x + z = 1 & \textbf{(5)}
\end{array}
$$

(3) $\boxed{\mu = -1}$

(1) $1 + 2\lambda x - 1 = 0 \implies 2\lambda x = 0$

(2) $2\lambda y = -1$

Since $2\lambda y = -1$, we know $\lambda \neq 0$ so $\boxed{x = 0}$

(4) $0^2 + y^2 = 2 \implies \boxed{y = \pm\sqrt{2}}$

(5) $0 + z = 1 \implies \boxed{z = 1}$

$$x = 0, y = \pm\sqrt{2}, z = 1$$

Thus, there are two critical points

$$(x, y, z) = (0, \sqrt{2}, 1) \text{ and } (0, -\sqrt{2}, 1)$$

$$f(0, \sqrt{2}, 1) = 1 + \sqrt{2} \text{ is a Relative Maximum}$$

$$f(0, -\sqrt{2}, 1) = 1 - \sqrt{2} \text{ is a Relative Minimum}$$

Note: $x^2 + y^2 = 2 \implies -\sqrt{2} \le x \le \sqrt{2}$ and $-\sqrt{2} \le y \le \sqrt{2}$

So x, y are bounded.

Since $x + z = 1$ and x is bounded, it follows that z is bounded.

Theorem

The Lagrange multiplier measures the rate of change of the extreme values of the objective function with respect to changes in the constraint constants.

Let's see why this is true. For simplicity, we'll examine functions of two variables. We set $F(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(g(x_1, x_2) - C)$ and find values x_1^*, x_2^*, λ^* so that $\nabla F(x_1^*, x_2^*, \lambda^*) = 0$. Thus

$$\begin{aligned} f_{x_1}(x_1^*, x_2^*) &= \lambda^* g_{x_1}(x_1^*, x_2^*) \\ f_{x_2}(x_1^*, x_2^*) &= \lambda^* g_{x_2}(x_1^*, x_2^*) \\ g(x_1^*, x_2^*) &= C \end{aligned}$$

and a maximum value $M = f(x_1^*, x_2^*)$.
Observe that $g(x_1^*, x_2^*) - C = 0$.

Now $\mathbf{x}^* = (x_1^*, x_2^*), \lambda^*,$ and M are all functions of C.
The derivative

$$\frac{df}{dC} f(\mathbf{x}^*(C)) \tag{1}$$

represents the rate of change in the optimal output with respect to a change of the constant C.
Corresponding to $\mathbf{x}^*(C)$ there is a value $\lambda = \lambda^*(C)$ giving a solution to the Lagrange multiplier problem;that is,

$$\begin{aligned} &\nabla f(\mathbf{x}^*(C)) = \lambda^*(C) \nabla g(x^*(C)) \text{ and} \\ &g(x^*(C)) = C \end{aligned} \tag{2}$$

We will show that

$$\lambda^*(C) = \frac{d}{dC} f(\mathbf{x}^*(C)) \tag{3}$$

which asserts that the Lagrange multiplier is the rate of change in the optimal output resulting from the change of the constant C. We present a derivation of this claim for two variables. The general case in n variables is the same, just replacing the sum of two terms by the sum of n terms.

$$\frac{d}{dC} f(\mathbf{x}^*(C)) = \frac{\partial f(\mathbf{x}^*(C))}{\partial x_1} \frac{dx_1^*}{dC}(C) + \frac{\partial f(\mathbf{x}^*(C))}{\partial x_2} \frac{dx_2^*}{dC}(C)$$

Because our values solve the Lagrange multiplier problem, we have

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*(C)) = \lambda^* \frac{\partial g}{\partial x_i}(\mathbf{x}^*(C)), \text{ all } i.$$

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$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*(C)) = \lambda^* \frac{\partial g}{\partial x_i}(\mathbf{x}^*(C)), \text{ all } i.$$

Substituting this result into the previous equation, we have
$\frac{d}{dC} f(\mathbf{x}^*(C))$

$$\begin{aligned}
&= \left[\lambda^*(C)\frac{\partial g}{\partial x_1}(\mathbf{x}^*(C))\right]\frac{dx_1^*}{dC}(C) + \left[\lambda^*(C)\frac{\partial g}{\partial x_2}(\mathbf{x}^*(C))\right]\frac{dx_2^*}{dC}(C) \\
&= \lambda^*(C)\left[\frac{\partial g}{\partial x_1}(\mathbf{x}^*(C))\frac{dx_1^*}{dC}(C) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(C))\frac{dx_2^*}{dC}(C)\right]
\end{aligned} \tag{4}$$

Since $C = g(\mathbf{x}^*(C))$ for all C, differentiation with respect to C gives

$$1 = \frac{d}{dC} g(\mathbf{x}^*(C)) = \frac{\partial g}{\partial x_1}(\mathbf{x}^*(C))\frac{dx_1^*}{dC}(C) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(C))\frac{dx_2^*}{dC}(C). \tag{5}$$

$$1 = \frac{d}{dC} g(\mathbf{x}^*(C)) = \frac{\partial g}{\partial x_1}(\mathbf{x}^*(C)) \frac{dx_1^*}{dC}(C) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(C)) \frac{dx_2^*}{dC}(C). \tag{6}$$

Replacing the right hand side by 1 in Equation (5.3) gives

$$\frac{d}{dC} f(\mathbf{x}^*(C)) = \lambda^*(C)\,[1] = \lambda^*(C)$$

which is our claim. In the economics perspective, if f is the profit function of the inputs, and C is the budget constraint, then the derivative is the rate of change of the profit from the change in the value of the inputs; the Lagrange multiplier is what economists call the *marginal profit of money*. They also use the term *shadow price* for the value of λ in the optimal solution of maximizing revenue subject to a budget constraint. The shadow price measures the money gained by loosening the constraint by a dollar or the loss of revenue if we tighten the constraint by a dollar.

Suppose Gradient of Our Function is 0 at Some Point.
How Do We Tell Whether It is:
Local Minimum.
Local Maximum.
Point of Inflection

Analogy From Calculus 1: Derivative is 0.
Is It Maximum, Minimum, Point of Inflection?

Second Derivative Test