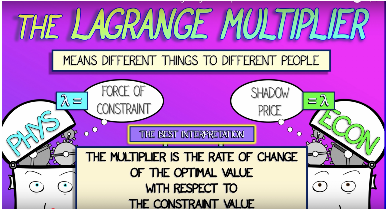
# MATH 223: Multivariable Calculus




Class 19: March 31, 2025

**Announcements**

Exam 2: Tonight at 7 PM

WNS 100: Last Name L – Z

WNS 101: Last Name A – K

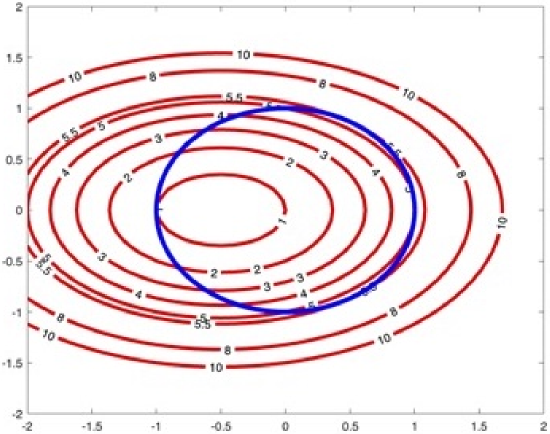
Lagrange Multiplier

Problem: Maximize $T(x, y) = 2x^2 + 4y^2 + 2x + 1$

subject to $x^2 + y^2 \leq 1$

**Contours for T(x,y)**

**Constraint Boundary**



## Look at a More General Problem
## Maximize f(xty) [Objective Function]
## Subject to g(x,y) = C [Constraint]

Examine Level Curves of $f$ $(f(x, y) = k$

Find intersection of constraint curve with level curve that has largest $k$.

This appears to occur at a point where the two curves are tangent to each other;

that is, gradient vectors point in the same direction.

## Theorem

*The Lagrange multiplier measures the rate of change of the extreme values of the objective function with respect to changes in the constraint constants.*

Let's see why this is true. For simplicity, we'll examine functions of two variables. We set $F(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(g(x_1, x_2) - C)$ and find values $x_1^*, x_2^*, \lambda^*$ so that $\nabla F(x_1^*, x_2^*, \lambda^*) = 0$. Thus

$$\begin{aligned} f_{x_1}(x_1^*, x_2^*) &= \lambda^* g_{x_1}(x_1^*, x_2^*) \\ f_{x_2}(x_1^*, x_2^*) &= \lambda^* g_{x_2}(x_1^*, x_2^*) \\ g(x_1^*, x_2^*) &= C \end{aligned}$$

and a maximum value $M = f(x_1^*, x_2^*)$.
Observe that $g(x_1^*, x_2^*) - C = 0$.

Now $\mathbf{x}^* = (x_1^*, x_2^*), \lambda^*,$ and $M$ are all functions of $C$.
The derivative

$$\frac{df}{dC} f(\mathbf{x}^*(C)) \tag{1}$$

represents the rate of change in the optimal output with respect to a change of the constant $C$.
Corresponding to $\mathbf{x}^*(C)$ there is a value $\lambda = \lambda^*(C)$ giving a solution to the Lagrange multiplier problem;that is,

$$\begin{aligned} \nabla f(\mathbf{x}^*(C)) &= \lambda^*(C) \nabla g(x^*(C)) \text{ and} \\ g(x^*(C)) &= C \end{aligned} \tag{2}$$

We will show that

$$\lambda^*(C) = \frac{d}{dC} f(\mathbf{x}^*(C)) \tag{3}$$

which asserts that the Lagrange multiplier is the rate of change in the optimal output resulting from the change of the constant $C$. We present a derivation of this claim for two variables. The general case in $n$ variables is the same, just replacing the sum of two terms by the sum of $n$ terms.

$$\frac{d}{dC} f(\mathbf{x}^*(C)) = \frac{\partial f(\mathbf{x}^*(C))}{\partial x_1} \frac{dx_1^*}{dC}(C) + \frac{\partial f(\mathbf{x}^*(C))}{\partial x_2} \frac{dx_2^*}{dC}(C)$$

Because our values solve the Lagrange multiplier problem, we have

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*(C)) = \lambda^* \frac{\partial g}{\partial x_i}(\mathbf{x}^*(C)), \text{ all } i.$$

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Substituting this result into the previous equation, we have $\frac{d}{dC} f(\mathbf{x}^*(C))$

$$\begin{aligned} &= \left[\lambda^*(C)\frac{\partial g}{\partial x_1}(\mathbf{x}^*(C))\right]\frac{dx_1^*}{dC}(C) + \left[\lambda^*(C)\frac{\partial g}{\partial x_2}(\mathbf{x}^*(C))\right]\frac{dx_2^*}{dC}(C) \\ &= \lambda^*(C)\left[\frac{\partial g}{\partial x_1}(\mathbf{x}^*(C))\frac{dx_1^*}{dC}(C) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(C))\frac{dx_2^*}{dC}(C)\right] \end{aligned} \tag{4}$$

Since $C = g(\mathbf{x}^*(C))$ for all $C$, differentiation with respect to $C$ gives

$$1 = \frac{d}{dC}g(\mathbf{x}^*(C)) = \frac{\partial g}{\partial x_1}(\mathbf{x}^*(C))\frac{dx_1^*}{dC}(C) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(C))\frac{dx_2^*}{dC}(C). \tag{5}$$

$$1 = \frac{d}{dC} g(\mathbf{x}^*(C)) = \frac{\partial g}{\partial x_1}(\mathbf{x}^*(C)) \frac{dx_1^*}{dC}(C) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(C)) \frac{dx_2^*}{dC}(C). \tag{6}$$

Replacing the right hand side by 1 in Equation (5.3) gives

$$\frac{d}{dC} f(\mathbf{x}^*(C)) = \lambda^*(C)\,[1] = \lambda^*(C)$$

which is our claim. In the economics perspective, if $f$ is the profit function of the inputs, and $C$ is the budget constraint, then the derivative is the rate of change of the profit from the change in the value of the inputs; the Lagrange multiplier is what economists call the *marginal profit of money*. They also use the term *shadow price* for the value of $\lambda$ in the optimal solution of maximizing revenue subject to a budget constraint. The shadow price measures the money gained by loosening the constraint by a dollar or the loss of revenue if we tighten the constraint by a dollar.

Example: Maximize production function $f(x,y) = x^{2/3}y^{1/3}$ subject to budget constraint $w = g(x,y) = px + qy$

Form

$$F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - w) = x^{2/3}y^{1/3} - \lambda(px + qy - w)$$

$$\begin{array}{ll} F_x(x,y,\lambda) = 0 & \frac{2}{3}x^{-1/3}y^{1/3} - \lambda p = 0 \\ F_y(x,y,\lambda) = 0 & \frac{1}{3}x^{2/3}y^{-2/3} - \lambda q = 0 \\ F_\lambda(x,y,\lambda) = 0 & px + qy = w \end{array}$$

$$\begin{gathered} \frac{2}{3}x^{-1/3}y^{1/3} = \lambda p \\ \frac{1}{3}x^{2/3}y^{-2/3} = \lambda q \\ px + qy = w \end{gathered}$$

$$\textbf{(1)}: \ \tfrac{2}{3}x^{-1/3}y^{1/3} = \lambda p$$
$$\textbf{(2)}: \ \tfrac{1}{3}x^{2/3}y^{-2/3} = \lambda q$$
$$\textbf{(3)}: \ px + qy = w$$

Multiply **(1)** by 3 and divide by $p$:

$$\tfrac{2}{p}x^{-1/3}y^{1/3} = 3\lambda$$

Multiply **(2)** by 3 and divide by $q$:

$$\tfrac{1}{q}x^{2/3}y^{-2/3} = 3\lambda$$

Thus $\frac{2}{p}x^{-1/3}y^{1/3} = \frac{1}{q}x^{2/3}y^{-2/3}$ which simplifies to

$$qy = \tfrac{1}{2}px$$

which makes **(3)**: $w = px + qy = px + \frac{1}{2}px = \frac{3}{2}px$

Hence $px^* = \frac{2}{3}w$ and $qy^* = \frac{1}{3}w$ so

$$x^* = \frac{2w}{3p} \text{ and } y^* = \frac{w}{3q}$$

To indicate the dependence on these optiimizing values on $w$, we write $x^*(w)$ and $y^*(w)$.

From **(1)** , we have $\lambda^*(w) = \frac{2}{3}(x^*)^{-1/3}(y^*)^{1/3}$

But $x^* = \frac{2w}{3p}$ and $y^* = \frac{w}{3q}$ so

$$\lambda^*(w) = \frac{2}{3p}\left(\frac{2w}{3p}\right)^{-1/3}\left(\frac{w}{3q}\right)^{1/3} = \frac{2^{2/3}}{3}\left(\frac{1}{p^2q}\right)^{1/3}$$

On the other hand, maximum value of the objective function $f$ is

$$\begin{aligned} f(x^*(w), y^*(w)) &= (x^*)^{2/3}(y^*)^{1/3} \\ &= \left(\frac{2w}{3p}\right)^{2/3}\left(\frac{w}{3q}\right)^{1/3} \\ &= \frac{2^{2/3}}{3}\left(\frac{1}{p^2q}\right)^{1/3} w = \lambda^*(w)w \end{aligned}$$

$$f(x^*(w), y^*(w)) = \lambda^*(w)w$$

so

$$\frac{d}{dw} f(x^*(w), y^*(w)) = \lambda^*(w)$$

Thus, the increase in the production at the point of maximization with respect to the increase in the value of the inputs equals to the Lagrange multiplier, i.e., the value of $\lambda^*$ represents the rate of change of the optimum value of $f$ as the value of the inputs increases.

Suppose Gradient of Our Function is 0 at Some Point.
How Do We Tell Whether It is:
Local Minimum.
Local Maximum.
Point of Inflection

Analogy From Calculus 1: Derivative is 0.
Is It Maximum, Minimum, Point of Inflection?

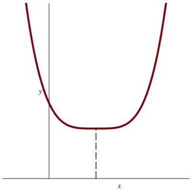
**Second Derivative Test**

Today:

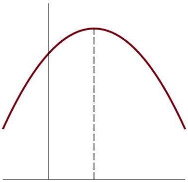
## Second Derivative Criteria

Classic Case: $f : \mathbb{R}^1 \to \mathbb{R}^1$

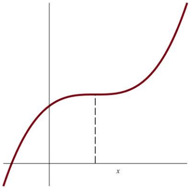
with $f'(x_0) = 0$



$f''(x_0) > 0$

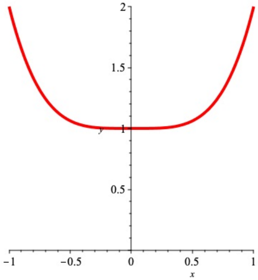


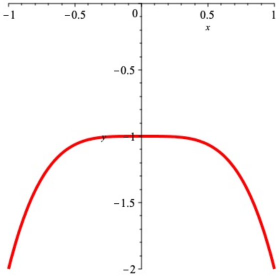
$f''(x_0) < 0$



$f''$ changes sign at $x_0$

# What Can We Conclude if $f''(x_0) = 0$?




$$f(x) = x^4$$
$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$
$$f''(0) = 0$$

$$f(x) = -x^4$$
$$f'(x) = -4x^3$$
$$f''(x) = -12x^2$$
$$f''(0) = 0$$

Second Derivative Test
For Real-Valued Functions of Several Variables

## Involves Second Order Partial Derivatives

**Definition**: If $f$ is a twice differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^1$, then the **Hessian Matrix** is the $n \times n$ matrix of second order partial derivatives of $f$

For example, if $f : \mathbb{R}^3 \to \mathbb{R}^1$ so $w = f(x, y, z)$, then the Hessian $\mathcal{H}$ for $f$ is

$$\mathcal{H}(f) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

Note that if $f$ is twice continuously differentiable, then the mixed partials are equal: $f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$ so the Hessian matrix is symmetric.


Otto Ludwig Hesse
April 22, 1811 – August 4, 1874

The Second Derivative Test
for real-valued functions of several variables
replaces the condition $f''(c)$ being positive
with the Hessian matrix being **positive definite**.
It similarly uses the **negative definite** character of the Hessian matrix in place of the negativity of the second derivative.