Notes on Assignment 19
Assignments 20 and 21
Curvilinear Coordinates

Announcements

Review Basic Theorems About Integration from Calculus I

Theorem

Second Derivative Test for Local Extrema.

Suppose $f : \mathbb{R}^n \to \mathbb{R}^1$ *has continuous third order partial derivatives on a neighborhood of* $\mathbf{x_0}$ *which is a critical point of* f *.*

IF the Hessian $\mathcal{H}$ *evaluated at* $\mathbf{x_0}$ *is positive definite, then* f *has a relative minimum at* $\mathbf{x_0}$*.*

If the Hessian is negative definite, then there is a relative maximum at the critical point.

Example: Our Temperature Function

$$T(x, y) = 2x^2 + 4y^2 + 2x + 1$$

Here $T_x(x, y) = 4x + 2$ and $T_y(x, y) = 8y$.

Thus, the Hessian Matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$$

whose eigenvalues are 4 and 8.

Both are positive so T has a minimum wherever the gradient is 0; that is, at (-1/2,0).

Example: $T(x, y) = x^2 - y^2$

$\nabla T(x, y) = (2x, -2y)$ so $\nabla T = \vec{0}$ at $(0, 0)$

Thus, the Hessian Matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

whose eigenvalues are 2 and -2.
Thus it is neither positive definite nor negative definite.

$\mathbf{x} \cdot \mathcal{H}\mathbf{x}$ can be positive ($\mathbf{x} = (1, 0)$) or negative ($\mathbf{x} = (0,1)$) so there is a **saddle point** at any point where ∇T is $\vec{0}$.

$\underline{\text{Example:}}$ $f(x,y) = x^3 - y^3 - 2xy$

Here $\nabla f = (3x^2 - 2y, -3y^2 - 2x)$

$\nabla f = \vec{0}$ when
$3x^2 = 2y$ and $3y^2 = -2x$

The first equation gives $9x^4 = 4y^2$ and the second yields $y^2 = -\frac{2}{3}x$
Thus $9x^4 = 4(-\frac{2}{3}x) = -\frac{8}{3}x$
so $9x^4 = -\frac{8}{3}x$ or $27x^4 + 8x = 0$; Hence $x(27x^3 + 8) = 0$
This gives two solutions: $x = 0, y = 0$ and $x = -\frac{2}{3}, y = \frac{2}{3}$

Two Critical Points: (0,0) and (-2/3, 2/3)

Example: $f(x,y) = x^3 - y^3 - 2xy$

$\nabla f = (3x^2 - 2y, -3y^2 - 2x)$

Two Critical Points: (0,0) and (-2/3, 2/3)

The Hessian Matrix is

$$\mathcal{H} = \begin{pmatrix} 6x & -2 \\ -2 & -6y \end{pmatrix}$$

At (-2/3,2/3), Hessian is

$$\mathcal{H} = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}$$

whose eigenvalues are -2 and -6, both negative.

Thus there is a relative maximum at (-2/3,2/3),/

Example: $f(x,y) = x^3 - y^3 - 2xy$
$\nabla f = (3x^2 - 2y, -3y^2 - 2x)$
Two Critical Points: (0,0) and (-2/3, 2/3)
The Hessian Matrix is

$$\mathcal{H} = \begin{pmatrix} 6x & -2 \\ -2 & -6y \end{pmatrix}$$

At (0,0), Hessian is

$$\mathcal{H} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

whose eigenvalues are -2 and $+2$.
. Thus there is a saddle point at (0,0),

More About Saddle Points

"Relative Maximum in One Direction, but Relative Minimum in Another Direction"

How Do We Find These Directions?

Look at the Eigenvectors!

Take our example $f(x,y) = x^3 - y^3 - 2xy$ at the origin.

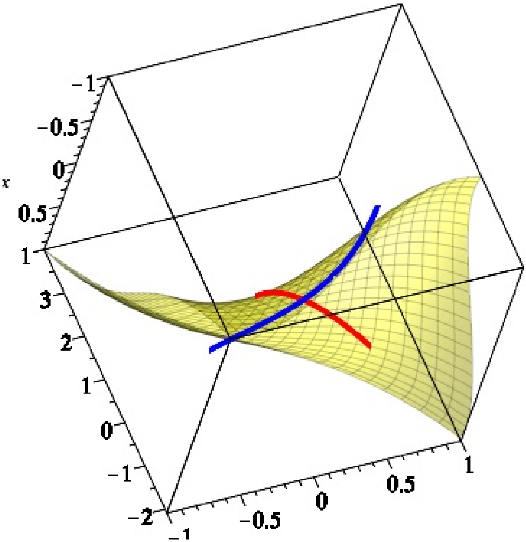
The eigenvalue -2 has eigenvector of the form (1,1)

Consider $f(x,x) = x^3 - x^3 - 2xx = -2x^2$ has relative maximum at $x = 0$

The eigenvalue $+2$ has eigenvector of the form (1,-1).

Consider $f(x,-x) = x^3 + x^3 + 2xx = 2x^3 + 2x^2 = 2x^2(1+x)$ has relative minimum at $x = 0$

Graph of $f(x, y) = x^3 - y^3 - 2xy$

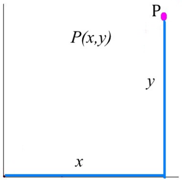


Next Time

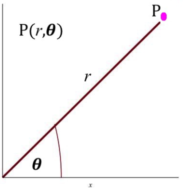
Alternative Coordinate Systems for 3-Space

Rectangular
Cylindrical
Spherical

Today:
Curvilinear Coordinates
Coordinate Systems in Plane and Space
Plane



Cartesian



Polar

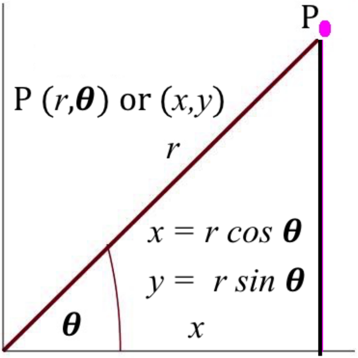
Newton(1671) [Not published until after death]
Jacob Bernoulli (1691)

In Polar Coordinates, Circles and Lines Through Origin Have Simple Equations:

Circle: $r = 4$

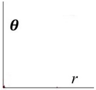
Line: $\theta = \pi/6$

Relationship Between Polar and Cartesian

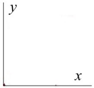


Linear Algebra Perspective

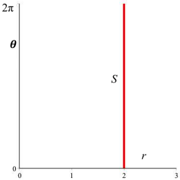
$$P\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ for } \begin{array}{c} 0 < r < \infty \\ 0 \le \theta \le 2\pi \end{array}$$

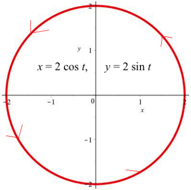


$$P : U^2 \to \mathbb{R}^2$$



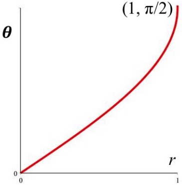
Example: P sends this line segment S to circle of radius 2 with center at origin.





$$[r,\theta] = [2,t], 0 \le t \le 2\pi$$

Example: $\theta = t, r = \sin t = \sin\theta$



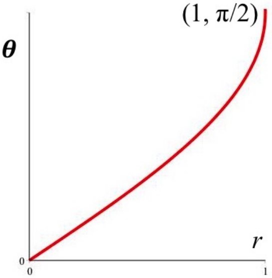
	$r = \sin t$	$\theta = t$
0	0	0
$\pi/6$	$1/2$	$\pi\ /6$
$\pi/4$	$\sqrt{2}/2$	$\pi/4$
$\pi/2$	1	$\pi/2$

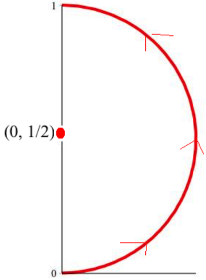
Then $x = r\cos\theta = \sin\theta\cos\theta$ and $y = r\sin\theta = \sin\theta\sin\theta$
So $x^2 = \sin^2\theta\cos^2\theta, y^2 = \sin^2\theta\sin^2\theta$
and then $x^2 + y^2 = \sin^2\theta(\cos^2\theta + \sin^2\theta) = \sin^2\theta \times 1 = \sin^2\theta = y$
Thus $x^2 + y^2 - y = 0$. Complete the square in y:
$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \implies x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$
which is the equation of a circle with center at (0,1/2) and radius 1/2.

$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \implies x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$$

which is the equation of a circle
with center at $(0,1/2)$ and radius $1/2$.

The image is the right half of the circle:





Think of P as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then

$$P' = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{pmatrix}$$

$$P' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \implies P'(\pi/6) = \begin{pmatrix} \sqrt{3}/2 & -1/4 \\ 1/2 & \sqrt{3}/4 \end{pmatrix}$$

$$\frac{\text{Previous Example}}{g(t) = [\sin t, t]}$$

$$\text{so } g'(t) = [\cos t, 1]$$

$$\text{At } t = \pi/6, g'(\pi/6) = \left[\sqrt{3}/2, 1\right]$$

$$g : \mathbb{R}^1 \to \mathbb{U}^2 \text{ and } P : \mathbb{U}^2 \to \mathbb{R}^2$$

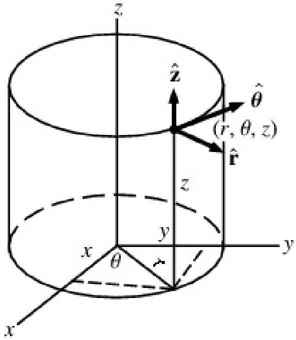
$$(P \circ g) : \mathbb{R}^1 \to \mathbb{R}^2$$

$$(P \circ g)' = P'(g) \cdot g'$$

$$\text{Evaluate at } \pi/6\text{: } \begin{pmatrix} \sqrt{3}/2 & -1/4 \\ 1/2 & \sqrt{3}/4 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

Coordinate Systems in 3-Space

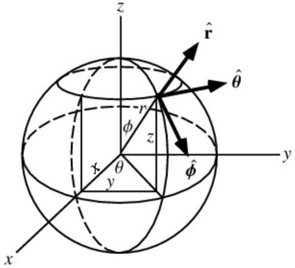
Cylindrical Coordinates: (r, θ, z).



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Coordinate Systems in 3-Space

Spherical Coordinates: $(\rho, \theta, \phi) = (r, \theta, \phi)$



$r =$ distance between origin and point

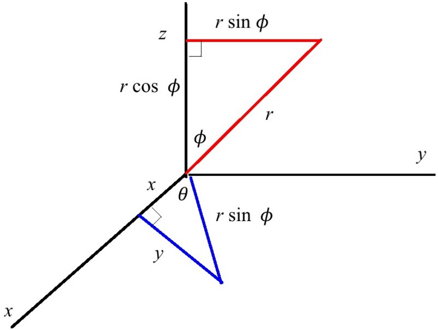
$\theta =$ project down to xy-plane

$\phi =$ rotation down from vertical axis

$r =$ distance between origin and point	$x = r \sin\phi \cos\theta$
$\theta =$ project down to xy-plane.	$y = r \sin\phi \sin\theta$
$\phi =$ rotation down from vertical axis	$z = r \cos\phi$

Some authors use ρ instead of r.

Converting from Spherical To Cartesian

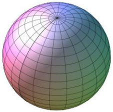


$$\cos\phi = \frac{z}{r} \implies z = r\cos\phi$$

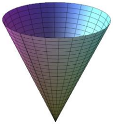
$$\cos\theta = \frac{x}{r\sin\phi} \implies x = r\sin\phi\cos\theta$$

$$\sin\theta = \frac{y}{r\sin\phi} \implies y = r\sin\phi\sin\theta$$

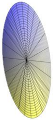
$$S : \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \to \begin{pmatrix} r \sin\phi \cos\theta \\ r \sin\phi \sin\theta \\ r \cos\phi \end{pmatrix}$$



$r =$ Constant
Sphere



$\phi =$ Constant
Cone



$\theta =$ Constant
Plane

Jacobian Matrices

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}$$