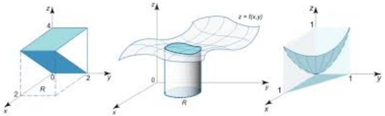
MATH 223: Multivariable Calculus



Class 22: April 7, 2025

Assignment 21

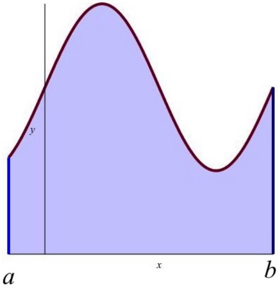
Announcements

Initial Responses on Location Problem: Friday

Theme For Rest of the Course:

INTEGRATION

Classic Integration I; $f : \mathbb{R}^1 \to \mathbb{R}^1$



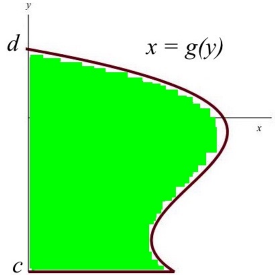
$$\int_a^b f(x)dx$$

Topic A: Iterated Integral

Setting: $f := \mathbb{R}^n \to \mathbb{R}^1$

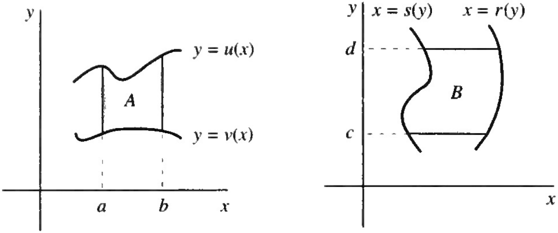
Initial Focus: $f := \mathbb{R}^2 \to \mathbb{R}^1$

Classic Integration 2



$$\int_c^d g(y)dy$$

Classic Integration



"Partial Integration": Notation

$$\int_a^b f(x,y)dy$$

Means Integrate With Respect to y, treating x as a constant

Some Examples of Partial Integration

$$\int_1^2 (x+y)dy = \left(xy + \frac{y^2}{2}\right)_{y=1}^{y=2} = (2x + \frac{4}{2}) - (x + \frac{1}{2}) = x + \frac{3}{2}$$

$$\int_0^3 xe^{xy}dy = (e^{xy})_{y=0}^{y=3} = e^{3x} - e^{x \cdot 0} = e^{3x} - e^0 = e^{3x} - 1$$

$$\int_3^4 x^2y^3dy = \left(x^2\frac{y^4}{4}\right)_{y=3}^{y=4} = x^2\left[\frac{4^4}{4} - \frac{3^4}{4}\right] = x^2\left[\frac{256-81}{4}\right] = \frac{175}{4}x^2$$

Note: All These Results are Functions of x

$$\int_c^d f(x,y)dy = F(x)$$

Here is an example involving integration with respect to **x**:

$$\int_3^4 x^2 y^3 dx = \frac{x^3}{3} y^3 \Big|_{x=3}^{x=4} = y^3 \left[\frac{4^3}{3} - \frac{3^3}{3} \right] = \frac{37}{3} y^3$$

$$\int_c^d f(x,y)dx \text{ is a function of } y, \text{ say } G(y)$$

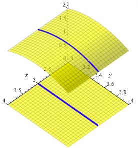
$$\int_c^d f(x,y)dy = F(x)$$

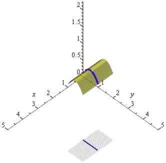
What is the meaning of $F(x)$?

Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be defined on the rectangle $a \le x \le b, c \le y \le d$ with $f(x,y) \ge 0$.

Fix x. Then the graph of $z = f(x,y)$ for $c \le y \le d$ is a curve and $F(x)$ is the area under this curve.

It is the area of a cross-section by a plane perpendicular to x-axis.





Adding up all these cross-sectional areas for all x:

$$\int_a^b F(x)dx = \int_a^b \int_c^d f(x,y)dydx = \int_a^b dx \int_c^d f(x,y)dy$$

This process is called **Iterated Integration**

Example

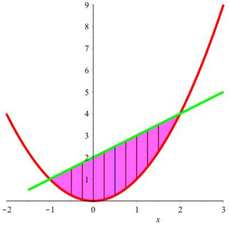
$$\int_0^1 \int_3^4 x^2 y^3 dydx = \int_0^1 \frac{175}{4} x^2 dx = \frac{175}{4} \frac{x^3}{3}\Big|_0^1 = \frac{175}{12}$$

Note: The final result is a **NUMBER**.
We could have integrated in the opposite order:

$$\int_3^4 \left(\int_0^1 x^2 y^3 dx \right) dy = \int_3^4 \frac{x^3}{3} y^3 \Big|_{x=0}^{x=1} dy = \int_3^4 \frac{y^3}{3} dy = \frac{y^4}{2} \Big|_{y=3}^{y=4} = \frac{175}{12}$$

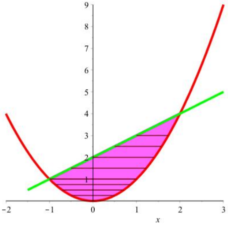
Example Let D be the region bounded by the parabola $y = x^2$ and the line $y = x + 2$ below and above by the graph of $z = f(x, y) \geq 0$ in D. Find the Volume of solid.

Vertical Strips



$$\int_{x=-1}^{x=2} \int_{y=x^2}^{y=x+2} f(x, y)dydx$$

Horizontal Strips



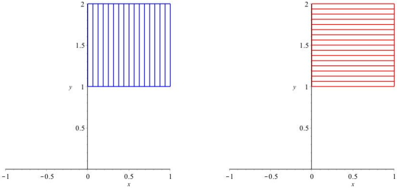
$$\int_{y=0}^{y=1} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} f(x, y)dxdy + \int_{y=1}^{y=4} \int_{x=y-2}^{x=\sqrt{y}} f(x, y)dxdy$$

Special Case: Integrating $f(x, y) = 1$ over a region in the plane gives the area of the region.

Order of Integration

An Example

Integrate $f(x, y) = xe^{xy}$ over $0 \le x \le 1, 1 \le y \le 2$



Method I:

$$\int_0^1 \left(\int_1^2 xe^{xy}\, dy \right) dx = \int_0^1 e^{xy} \Big|_{y=1}^{y=2} dx = \int_0^1 e^{2x} - e^x dx$$

$$= (\frac{e^2}{2} - e^1) - (\frac{e^0}{2} - e^0) = \frac{e^2}{2} - e - \frac{1}{2} + 1 = \frac{e^2}{2} - e + \frac{1}{2}$$

Method II: Reverse the Order of Integration

$$\int_{y=1}^{y=2} \left(\int_0^1 xe^{xy}\, dx \right) dy$$

The inside integration (with respect to x) requires Integration by parts:

Let $u = x$ and $dv = e^{xy}\, dx$. Then $du = dx$ and $v = \frac{1}{y} e^{yx}$

so $\int xe^{xy}\, dx = \frac{x}{y} e^{xy} - \int \frac{1}{y} e^{xy}\, dx = \frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy}$

Thus $\int_0^1 xe^{xy}\, dx = \left(\frac{x}{y} e^{xy} - \frac{1}{y^2} e^{xy} \right) \Big|_{x=0}^{x=1} = \frac{1}{y} e^y - 0 - \frac{1}{y^2} e^y + \frac{1}{y^2}$

Our integral problem becomes $\int_{y=1}^{y=2} \left[\frac{1}{y} e^y - \frac{1}{y^2} e^y + \frac{1}{y^2} \right] dy$

which is an extremely hard integral to find!