MATH 223: Multivariable Calculus

Class 23: April 9, 2025

Department of Mathematics and Statistics

Pre-registration Dessert Social

Monday, 4/14 | 3:00-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2025**? Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2025 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

A special note to our Math & Stat Majors: we will take class year pictures of our majors at **3:30pm**. These pictures will be displayed in the Math & Stat hallway alongside your name. Those who cannot or do not want to be in the picture will still have their names displayed.

Multiple Integrals
Assignment 21

Activity	Date	Approximate Weight
Exam 3	April 28	20%
Project	May 12	10%
Final Exam	May 15	30%

The Week Ahead:

Iterated Integral (Last Time)
Definition of Multiple Integrals
Properties of the Integral
Change of Variable

Instances of the Integral: I
The Classic Case

$$f := \mathbb{R}^1 \to \mathbb{R}^1$$

Definite Integral $\int_a^b f(x)dx$

Indefinite Integral $\int f(x)dx$

Instances of the Integral: II

Vector-Valued Functions of a Real Variable

$$f := \mathbb{R}^1 \to \mathbb{R}^n$$

$$f(t) = [f_1(t), f_2(t), ..., f_n(t)]$$

$$\int f(t) = \left[\int f_1(t), \int f_2(t), ..., \int f_n(t)\right]$$

Instances of the Integral: III

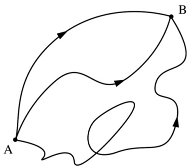
Line Integral = Path Integral

(Will Study in Chapter 7)

γ is graph of $g : \mathbb{R}^1 \to \mathbb{R}^n, a \leq t \leq b$

Force Field $F : \mathbb{R}^n \to \mathbb{R}^n$

$$\int_\gamma F = \int_a^b F(g(t)) \cdot g'(t) dt$$

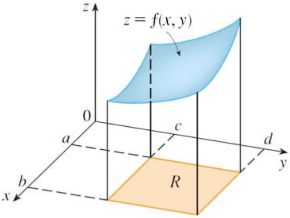


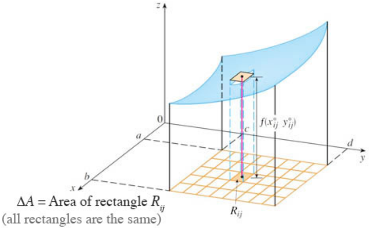
Instances of the Integral: IV

Iterated Integral

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$

$$\int_a^b \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy \right) dx$$





Instances of the Integral: V

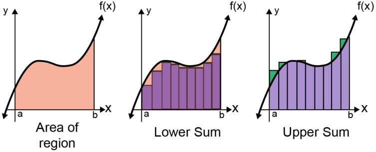
The Multiple Integral

A Generalization of Situation 1

$$\int_a^b f(x)dx = \lim_{max(\Delta(x_i)\to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

$$\Delta x_i = \text{ length of ith subdivision}$$

$$x_i = \text{ any point in ith subinterval}$$



Instances of the Integral: V

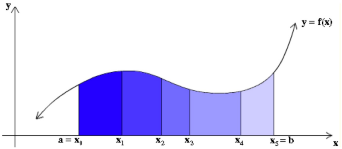
Unequal Divisions

$$\int_a^b f(x)dx = \lim_{max(\Delta(x_i)\to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

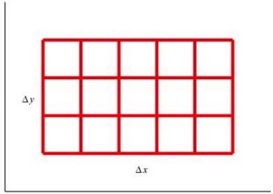
$$\Delta x_i = \text{ length of ith subdivision}$$

$$x_i = \text{ any point in ith subinterval}$$

Riemann Sums of Unequal Length Subintervals



First Extension: $f:\mathbb{R}^2 \to \mathbb{R}^1$ on a rectangle $\mathcal{R}$



Cover $\mathcal{R}$ with a grid G of horizontal and vertical lines

$mesh(G) = m(G)$=maximum length of edge of interval in grid.

Number Rectangles $R_1, R_2, ..., R_k$ with area of Rectangle R_i denoted by $A(R_i)$.

Pick a point $\vec{x_i}$ in R_i.

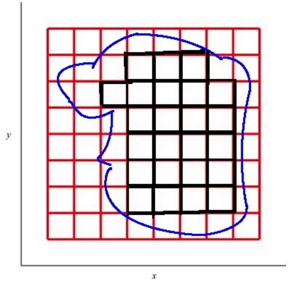
$$\text{Form } \sum_{i=1}^{k} f(\vec{x_i})A(R_i); \text{ Take } \lim_{m(G)\to 0} \sum_{i=1}^{k} f(\vec{x_i})A(R_i)$$

The limit is the integral of f over $\mathcal{R}$ and is denoted $\int_{\mathcal{R}} f dA$

Second Extension: $f : \mathbb{R}^2 \to \mathbb{R}^1$ on a BOUNDED set $\mathcal{B}$

Cover $\mathcal{B}$ with a grid G of horizontal and vertical lines

Let $R_1, R_2, ..., R_k$ be all bounded rectangles formed by G that lie **inside** $\mathcal{B}$.



Choose $\vec{x_i}$ in R_i.

Take $\lim_{m(G)\to 0} \sum_{i=1}^{k} f(\vec{x_i})A(R_i)$

The limit is the integral of f over $\mathcal{B}$ and is denoted $\int_{\mathcal{B}} fdV$

Theorem: If $\int_{\mathcal{B}} f dV$ exists and iterated integrals exist for some orders of partial integration, then all of these integrals are equal.

Proof:
Apostol, *Mathematical Analysis*
Sprivak, *Calculus on Manifolds*

When Does The Integral Exist?

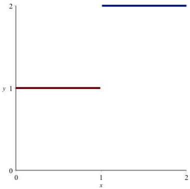
Idea: f does not have too many points of discontinuity

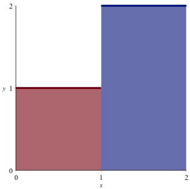
Definition: A set S has **zero content** if $\int_S 1 dV = 0$.

Theorem: Let $\mathcal{B}$ be a bounded set in $\mathbb{R}^n$ whose boundary has zero content. Let f be a bounded function bounded on $\mathcal{B}$. If f is continuous on $\mathcal{B}$ except perhaps on a set of zero content, then $\int_{\mathcal{B}} f dV$ exists.

Example from Calculus 1

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 2 & 1 < x \leq 2 \end{cases}$$





$$\int_1^2 f(x)dx = 3$$

Generalize For a Function $f : \mathbb{R}^n \to \mathbb{R}^1$

Coordinate Rectangle in $\mathbb{R}^n$

$\mathcal{R} = \{(x_1, x_2, ..., x_n) : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, ..., a_n \leq x_n \leq b_n\}$

Volume or Content of $\mathcal{R}$

$V(\mathcal{R}) = (b_1 - a_a)(b_2 - a_2)...(b_n - a_n)$

Grid: a finite set of $n-1$ =dimensional planes in $\mathbb{R}$ parallel to the coordinate planes.

G divides $\mathbb{R}$ into a finite number of bounded "rectangles' $\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_k$ and possibly other unbounded "rectangles,

The mesh $m(G)$ of a grid $=$ maximum "length" of a side of the rectangles $\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_k$

A set $\mathcal{B}$ is bounded if it can be covered by a grid.

$$\text{Then } \int_{\mathcal{B}} f dV = \lim_{m(G) \to 0} \sum_{i=1}^{k} f(\vec{x_i}) V(R_i)$$

if this limit exists. for all grids and all choices of $\vec{x_i}$ in R_i.

$$\text{Then } \int_{\mathcal{B}} f dV = \lim_{m(G)\to 0} \sum_{i=1}^{k} f(\vec{x_i}) V(R_i)$$

if this limit exists. for all grids and all choices of $\vec{x_i}$ in R_i.

$$\text{Content of } \mathcal{B} = \int_{\mathcal{B}} 1 dV = \begin{cases} \text{Length of } \mathcal{B} & \text{if } \mathcal{B} \subset R^1, \\ \text{Area of } \mathcal{B} & \text{if } \mathcal{B} \subset R^2 \\ \text{Volume of } \mathcal{B} & \text{if } \mathcal{B} \subset R^3 \end{cases}$$

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$ **using the definition**.

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

For each $n = 1, 2, ...$ consider the Grid G_n consisting of
the vertical lines $x = \frac{i}{n}, i = 0, 1, ..., n$ and
the horizontal lines $y = \frac{j}{n}, j = 0, 1, ..., 3n$

Then mesh of $G_n = \frac{1}{n}$ and Area of Rectangle $R_{ij} = \frac{1}{n^2}$

Riemann sum is $\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[\left(\frac{i}{n}\right)^2+5\left(\frac{j}{n}\right)\right]\right)A(R_{ij})$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{3n}\left(\frac{i}{n}\right)^2+\sum_{i=1}^{n}\sum_{j=1}^{3n}5\left(\frac{j}{n}\right)\right]$$

$$= \frac{1}{n^2}\left[3n\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2+n\sum_{j=1}^{3n}\frac{5j}{n}\right]$$

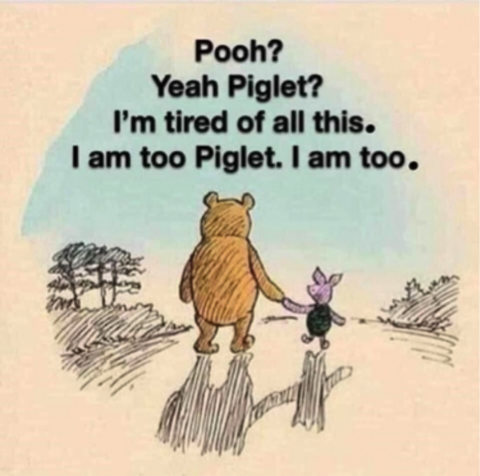
$$= \frac{1}{n^2}\left[\frac{3n}{n^2}\sum_{i=1}^{n}i^2+\frac{5n}{n}\sum_{j=1}^{3n}j\right]$$

$$= \frac{1}{n^2}\left[\frac{3}{n}\frac{n(n+1)(2n+1)}{6}+5\frac{(3n)(3n+1)}{2}\right]$$

Riemann sum is $\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[(\frac{i}{n})^2+5(\frac{j}{n})\right]\right)A(R_{ij})$

$$= \frac{1}{n^2}\left[\frac{1}{2}(n+1)(2n+1)+\frac{15}{2}n(3n+1)\right]$$
$$= \frac{1}{2}\left[(1+\frac{1}{n})(2+\frac{1}{n})\right]+\frac{15}{2}\left[3+\frac{1}{n}\right]$$

Hence $\lim_{n\to\infty}=\frac{1}{2}(2)+\frac{15}{2}(3)=\frac{47}{2}$



There Must Be a Better Way!

Evaluate As Iterated Integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2 + 5y)dydx$$

$$= \int_{x=0}^{x=1} \left[x^2 y + \frac{5}{3} y^2 \right]_{y=0}^{y=3} dx$$

$$= \int_0^1 3x^2 + \frac{45}{2} dx = \left[x^3 + \frac{45}{2} x \right]_0^1 = (1 + \frac{45}{2}) - (0 + 0) = \frac{47}{2}$$