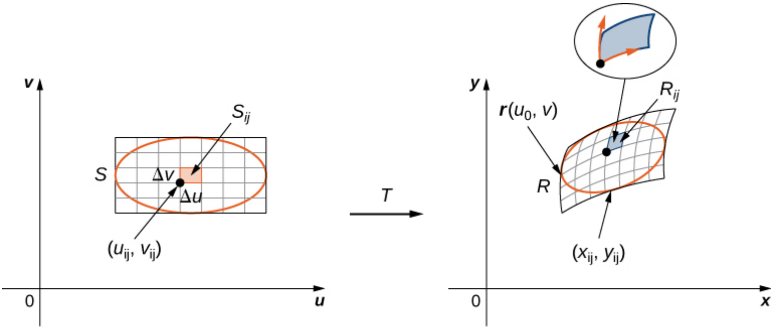
MATH 223: Multivariable Calculus



Class 25: Wednesday, April 16, 2025

Notes on Assignment 22
Assignment 23
Jacobi's Theorem on Change of Variable

Announcements

Review Improper Integrals:

$$\int_1^{\infty} \frac{1}{x^n}\, dx$$

Progress Report on Location Problem:

Due By Friday, April 18

Should Have Explicit Function To Minimize With Full Rationale

Upcoming Topics:

Change of Variable

Improper Integrals

Application to Probability

Change of Variable aka Method of Substitution

A common technique in the evaluation of integrals is to make a change of variable in the hopes of simplifying the problem of determining an antiderivatives

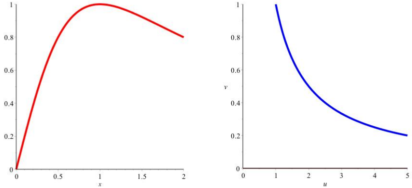
Example: Evaluate $\int_{x=0}^{x=2} \frac{2x}{1+x^2} dx$

$$\begin{array}{l|l} \text{Let } u = 1 + x^2 & x = 0 \to u = 1 + 0^2 = 1 \\ \text{The } du = 2x dx & x = 2 \to u = 1 + 2^2 = 5 \end{array}$$

$$\int_{x=0}^{x=2} \frac{2x}{1+x^2} dx = \int_{u=1}^{u=5} \frac{1}{u} du = \ln 5 - \ln 1 = \ln 5$$

$$\int_{x=0}^{x=2} \frac{2x}{1+x^2}\,dx = \int_{u=1}^{u=5} \frac{1}{u}\,du$$

Let's look at what is happening geometrically:



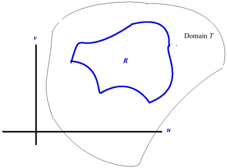
Not only does the function change, but also the region of integration.

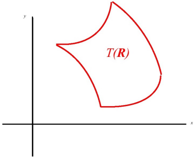
The region of integration changes from an interval of length 2 to an interval of length 4.

The interval also moves to a new location.

In computing mutltiple integrals, the corresponding change in the region may be more complicated.

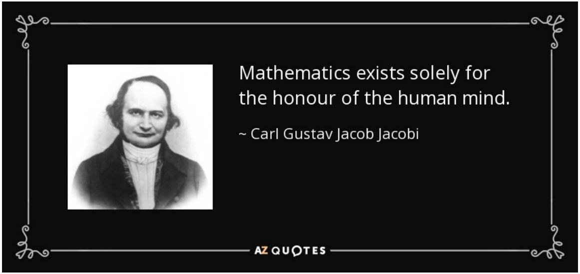
By a **change of variable**, we will mean a vector function T from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is convenient to use different letters to denote the spaces; e.g, $T : \mathbb{U}^n \to \mathbb{R}^n$





Carl Gustav Jacob Jacobi

December 10, 1804 – February 18, 1851



For further information see his Biography

Jacobi's Theorem

Let $\mathcal{R}$ be a set in $\mathbb{U}^n$ and $T(\mathcal{R})$ its image under T; that is,

$$T(\mathcal{R}) = \{T(\vec{u}) : \vec{u} \text{ is in } \mathcal{R}\}$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}^1$ is a real-valued function.

Then, under suitable conditions,

$$\int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u}))|det T'(\vec{u})| dV_{\vec{u}}$$

- T is continuous differentiable
- Boundary of $\mathcal{R}$ is finitely many smooth curves
- T is one-to-one on interior of $\mathcal{R}$
- The Jacobian Determinant det T' is non zero on interior of $\mathcal{R}$.
- The function f is bounded and continuous on $T(\mathcal{R})$

$$\int_{T(\mathcal{R})} f(\vec{x})dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u})|detT'(\vec{u}))|dV_{\vec{u}}$$

In our example: $u = 1 + x^2$ so $x = \sqrt{u-1}$
Thus $T(u) = \sqrt{u-1} = (u-1)^{1/2}$ so
$T'(u) = \frac{1}{2}(u-1)^{-1/2} = \frac{1}{2\sqrt{u-1}}$
$\int_0^2 \frac{2x}{1+x^2}dx = \int_{T(\mathcal{R})} f(\vec{x})dV_{\vec{x}} = \int_1^5 f(T(u)|detT'(u)|du$

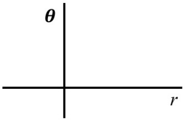
$$\text{Now } f(T(\vec{u}) = \frac{2T(u)}{1+(T(u))^2} = \frac{2\sqrt{u-1}}{1+u-1} = \frac{2\sqrt{u-1}}{u}$$

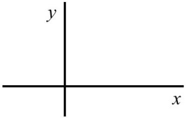
$$\det T'(u) = \left|\frac{1}{2\sqrt{u-1}}\right| = \frac{1}{2\sqrt{u-1}} \text{ so } f(T(\vec{u})\det T'(u) = \frac{1}{u}$$

so $\int_0^2 \frac{2x}{1+x^2}dx = \int_1^5 \frac{2\sqrt{u-1}}{u}\frac{1}{2\sqrt{u-1}}du = \int_1^5 \frac{1}{u}du$

Example: **Polar Coordinate Change of Variable**

$$\mathcal{U}^2 \qquad T \rightarrow \qquad \mathcal{R}^2$$





$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

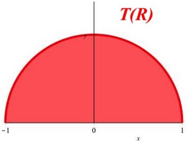
$$T' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \text{ so } \det T' = r\cos^2\theta + r\sin^2\theta = r$$

Thus $\int_{T(R)} f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$

$$\int_{T(R)} f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Example: $f(x,y) = x^2 + y^2$

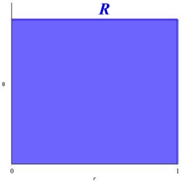
$T(R) =$ Half Disk $= \{(x,y) : -1 \le x \le 1, 0 \le y \le \sqrt{1-x^2}\}$



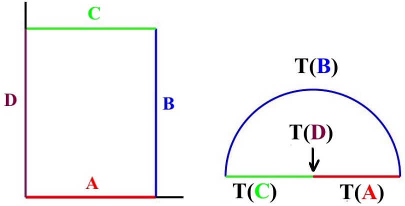
$$I = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$$

Describe Region in Polar Coordinates: $0 \le r \le 1, 0 \le \theta \le \pi$

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^{1} r^2\, r\,dr\,d\theta = \int_{\theta=0}^{\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \int_{\theta=0}^{\pi} \frac{1}{4} d\theta = \frac{\pi}{4}$$



Look At This Transformation More Closely



$A: 0 \leq r \leq 1, \theta = 0$

$x = r\cos\theta = r\cos 0 = r$

$y = r\sin\theta = r\sin 0 = 0$

$B: r = 1, 0 \leq \theta \leq \pi$

$x = r\cos\theta = \cos\theta$

$y = r\sin\theta = \sin\theta$

$C: 0 \leq r \leq 1, \theta = \pi$

$x = r\cos\theta = r\cos\pi = -r$

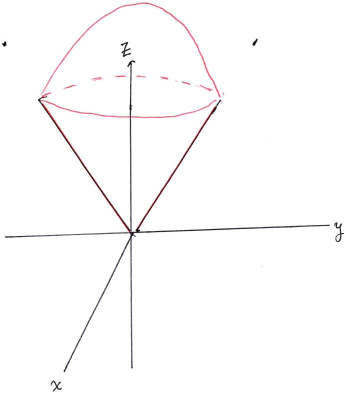
$y = r\sin\theta = r\sin\pi = 0$

$D: r = 0, 0 \leq \theta \leq \pi$

$x = r\cos\theta = 0$

$y = r\sin\theta = 0$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$
where C is the ice cream cone
$\{(x,y,z) : x^2+y^2+z^2 \leq 1, x^2+y^2 \leq \frac{z^2}{3}, z \geq 0\}$



Example: **Spherical Coordinates**

$$x = r\sin\phi\cos\theta \qquad T : (r,\phi,\theta) \to (x,y,z)$$
$$y = r\sin\phi\sin\theta \qquad \det\ T' = r^2\sin\phi$$
$$z = r\cos\phi$$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$

where C is the ice cream cone

$$\{(x,y,z) : x^2+y^2+z^2 \le 1, x^2+y^2 \le \tfrac{z^2}{3}, z \ge 0\}$$

$z \ge 0$ implies $\phi \le \frac{\pi}{2}$

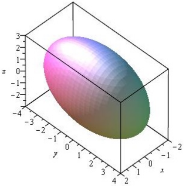
$x^2+y^2+z^2 \le 1$ implies $r \le 1$

$x^2+y^2 \le \frac{z^2}{3}$ implies $r^2\sin^2\phi \le \frac{r^2\cos^2\phi}{3}$

implies $\tan^2\phi \le \frac{1}{3}$ implies $\phi \le \frac{\pi}{6}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6}\int_{r=0}^{1}\sqrt{r^2}\,r^2\sin\phi\,dr\,d\phi\,d\theta$$

Example: Evaluate $\iiint_D z^2\, dV$ where D is the interior of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$



STEP 1: Let $u = \frac{x}{2}, v = \frac{y}{4}, w = \frac{z}{3}$.

Equation of the ellipsoid becomes $u^2 + v^2 + w^2 = 1$ (unit sphere)

So $x = 2u, y = 4v, z = 3w$ gives $T(u, v, w) = (2u, 4v, 3w)$ and

$$T' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ so det } T' = 2 \times 4 \times 3 = 24$$

Thus $\iiint_D z^2 = \iiint (3w)^2 (24)\, du\, dv\, dw = 216 \iiint w^2\, du\, dv\, dw$

STEP 2: Switch to Spherical Coordinates:

$$u = r\sin\phi\cos\theta, v = r\sin\phi\sin\theta, w = r\cos\phi$$

$$216\iiint w^2\,du\,dv\,dw = 216\iiint (r\cos\phi)^2 r^2 \sin\phi\,dr\,d\phi\,d\theta$$

$$= 216\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4\cos^2\phi\sin\phi\,dr\,d\phi\,d\theta$$

$$= (216)(2\pi)\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4\cos^2\phi\sin\phi\,dr\,d\phi$$

$$= (216)(2\pi)\frac{1}{5}\int_{\phi=0}^{\pi}\cos^2\phi\sin\phi\,d\phi$$

$$= \frac{(216)(2\pi)}{5}\left[-\frac{\cos^3\phi}{3}\right]_{\phi=0}^{\pi} = \frac{(216)(2\pi)}{5}\frac{2}{3} = \frac{288\pi}{5}$$