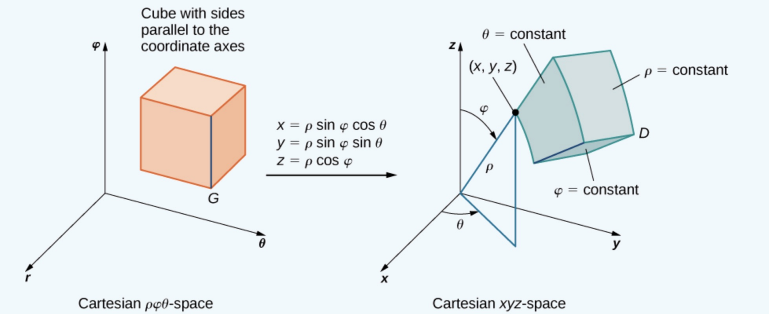
MATH 223: Multivariable Calculus



Class 26: Friday, April 18, 2025

Notes on Assignment 24

Assignment 25

Improper Integrals and Probability Density Functions

Progress Report on Location Problem:

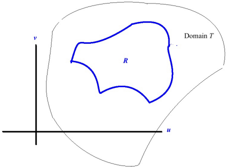
Should Have Explicit Function To Minimize With Full Rationale

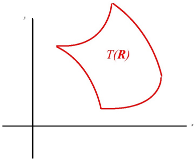
Announcements

This Week:
Properties of Integral
Leibniz Rule
Change of Variable
Improper Integrals
Application to Probability

In computing mutltiple integrals, the corresponding change in the region may be more complicated.

By a **change of variable**, we will mean a vector function T from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is convenient to use different letters to denote the spaces; e.g, $T : \mathbb{U}^n \to \mathbb{R}^n$





Jacobi's Theorem

Let $\mathcal{R}$ be a set in $\mathbb{U}^n$ and $T(\mathcal{R})$ its image under T; that is,

$$T(\mathcal{R}) = \{T(\vec{u}) : \vec{u} \text{ is in } \mathcal{R}\}$$

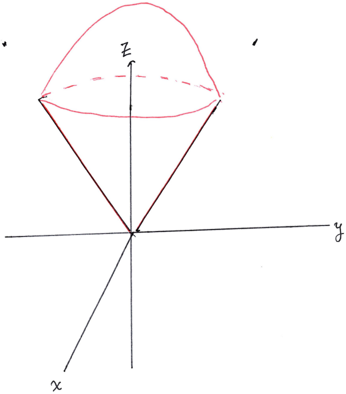
Suppose $f : \mathbb{R}^n \to \mathbb{R}^1$ is a real-valued function.

Then, under suitable conditions,

$$\int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u}))|det T'(\vec{u})| dV_{\vec{u}}$$

- T is continuous differentiable
- Boundary of $\mathcal{R}$ is finitely many smooth curves
- T is one-to-one on interior of $\mathcal{R}$
- The Jacobian Determinant det T' is non zero on interior of $\mathcal{R}$.
- The function f is bounded and continuous on $T(\mathcal{R})$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$
where C is the ice cream cone
$\{(x,y,z) : x^2+y^2+z^2 \le 1, x^2+y^2 \le \frac{z^2}{3}, z \ge 0\}$



Example: Spherical Coordinates

$$x = r\sin\phi\cos\theta \qquad T : (r,\phi,\theta) \to (x,y,z)$$
$$y = r\sin\phi\sin\theta \qquad \det\ T' = r^2\sin\phi$$
$$z = r\cos\phi$$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$

where C is the ice cream cone

$$\{(x,y,z) : x^2+y^2+z^2 \le 1, x^2+y^2 \le \tfrac{z^2}{3}, z \ge 0\}$$

$z \ge 0$ implies $\phi \le \frac{\pi}{2}$

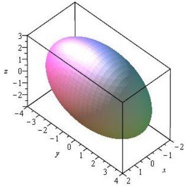
$x^2+y^2+z^2 \le 1$ implies $r \le 1$

$x^2+y^2 \le \frac{z^2}{3}$ implies $r^2\sin^2\phi \le \frac{r^2\cos^2\phi}{3}$

implies $\tan^2\phi \le \frac{1}{3}$ implies $\phi \le \frac{\pi}{6}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6}\int_{r=0}^{1} \sqrt{r^2}\,r^2\sin\phi\,dr\,d\phi\,d\theta$$

Example: Evaluate $\iiint_D z^2 \, dV$ where D is the interior of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$



STEP 1: Let $u = \frac{x}{2}, v = \frac{y}{4}, w = \frac{z}{3}$.

Equation of the ellipsoid becomes $u^2 + v^2 + w^2 = 1$ (unit sphere)

So $x = 2u, y = 4v, z = 3w$ gives $T(u, v, w) = (2u, 4v, 3w)$ and

$$T' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ so det } T' = 2 \times 4 \times 3 = 24$$

Thus $\iiint_D z^2 = \iiint (3w)^2 (24) \, du \, dv \, dw = 216 \iiint w^2 \, du \, dv \, dw$

STEP 2: Switch to Spherical Coordinates:

$u = r \sin\phi \cos\theta, v = r \sin\phi \sin\theta, w = r \cos\phi$

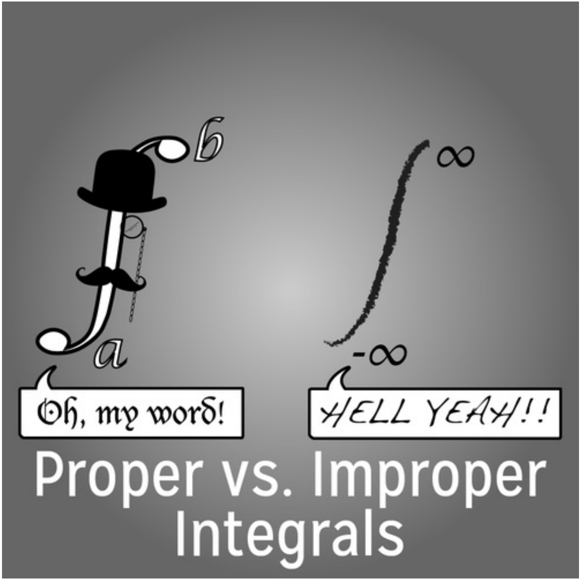
$$216 \iiint w^2 \, du \, dv \, dw = 216 \iiint (r\cos\phi)^2 r^2 \sin\phi \, dr \, d\phi \, d\theta$$

$$= 216 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=0}^{1} r^4 \cos^2\phi \sin\phi \, dr \, d\phi \, d\theta$$

$$= (216)(2\pi) \int_{\phi=0}^{\pi} \int_{r=0}^{1} r^4 \cos^2\phi \sin\phi \, dr \, d\phi$$

$$= (216)(2\pi)\frac{1}{5} \int_{\phi=0}^{\pi} \cos^2\phi \sin\phi \, d\phi$$

$$= \frac{(216)(2\pi)}{5} \left[-\frac{\cos^3\phi}{3} \right]_{\phi=0}^{\pi} = \frac{(216)(2\pi)}{5} \frac{2}{3} = \frac{288\pi}{5}$$


b
a
∞
-∞
Oh, my word!
HELL YEAH!!
Proper vs. Improper
Integrals

Improper Integrals

Setting $\int_{\mathcal{B}} f\, dV$ where $\mathcal{B}$ is a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^1$

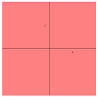
Two Types:

(I): $\mathcal{B}$ is unbounded

(II) $\mathcal{B}$ is bounded but f is unbounded

Type I Examples

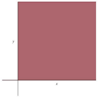
$\mathcal{B} = \mathbb{R}^2$



$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx$

$\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f^*(r, \theta) r\, d\theta\, dr$

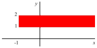
$\mathcal{B}$ = First Quadrant



$\int_0^{\infty} \int_0^{\infty} f(x, y)\, dy\, dx$

$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} f^*(r, \theta) r\, d\theta\, dr$

$\mathcal{B}$ is infinite strip



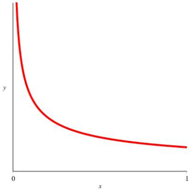
. $\int_{-1}^{\infty} \int_1^2 f(x, y)\, dy\, dx$

$$\int_{-1}^{\infty} \int_1^2 f(x, y)\, dy\, dx = \lim_{b \to \infty} \int_{-1}^{b} \int_1^2 f(x, y)\, dy\, dx$$

Type II Examples

Classic Case

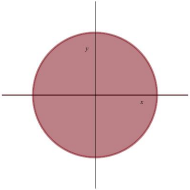
$$I = \int_0^1 \frac{1}{\sqrt{x}}\, dx$$



$$I = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \left[2\sqrt{x} \right]_a^1 = \lim_{a \to 0^+} \left[2 - 2\sqrt{a} \right] = 2$$

Type II Examples

In $\mathbb{R}^2, f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ on unit disk



In Polar Coordinates:

$$\int_0^1 \int_0^{2\pi} \frac{1}{r} r \, d\theta \, dr = \lim_{a \to 0^+} \int_a^1 \int_0^{2\pi} d\theta \, dr = \lim_{a \to 0^+} \int_a^1 2\pi dr$$

$$= \lim_{a \to 0^+} (2\pi - 2\pi a) = 2\pi$$


ONE DOES NOT SIMPLY
WRITE AN IMPROPER INTEGRAL
WITHOUT THE LIMIT

Improper Integrals

Let $\{B_\delta\}$ be a family of bounded sets B_δ that expands to cover all of the set B. We say $\int_B f(\mathbf{x})dV$ is defined as an **improper integral** if the limit

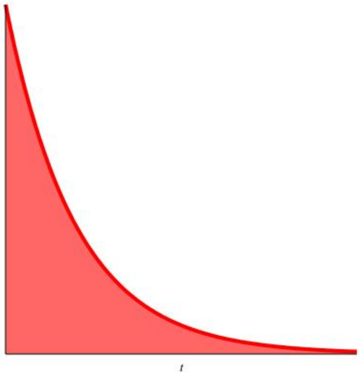
$$\int_B f(\mathbf{x})dV = \lim \int_{B_\delta} f(\mathbf{x})\,dV$$ is finite and independent of the family $\{B_\delta\}$ used to define it. If the limit exists (as a finite number), we say that the improper integral **converges** to that value. If the limit fails to exist, we say the improper integral **diverges.**

An Important Example:

Exponential Probability Density Function

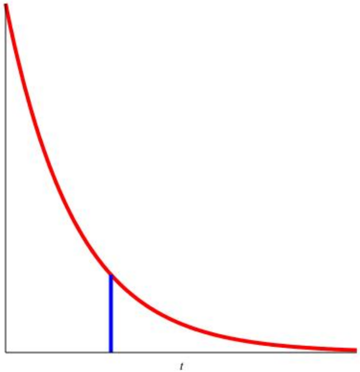
$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\Big|_{x=0}^{b}\right]$$

$$= \lim_{b\to\infty}\left[-e^{-b} - (-e^0)\right] = \lim_{b\to\infty}\left[1 - \frac{1}{e^b}\right] = 1$$



Exponential Probability Density Function

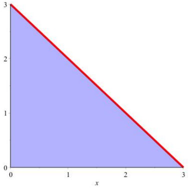
Probability(Light Bulb Burns Out in $\leq x$ months) = $\int_0^x e^{-t}\, dt = 1 - e^{-x}$



x	$\int_0^x e^{-t} dt$	Prob(Bulb Lasts More than x months)
1	.632	.368
2	.865	.135
3	.950	.050
4	.982	.018

Suppose You Buy 2 Light Bulbs

What Is The Probability They Will Provide At Least 3 Months of Service?



$$\text{Prob}(x + y > 3) = 1 \text{ - } \text{Prob}(x + y \leq 3)$$

$$= 1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$$

$$\text{Evaluate } 1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$$

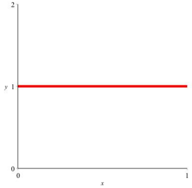
$$\begin{aligned}
&= 1 - \int_0^3 e^{-x} \left[-e^{-y} \Big|_{y=0}^{3-x} \right] dx \\
&= 1 - \int_0^3 e^{-x} \left[-e^{3-x} + 1 \right] dx \\
&= 1 - \int_0^3 (e^{-x} - e^{-3}) \, dx \\
&= 1 - \left[-e^{-x} - e^{-3} x \right]_{x=0}^{3}
\end{aligned}$$

$$= 1 - \left[-e^{-3} - 3e^{-3} + 1 + 0 \right] = 1 - \left[1 - \frac{4}{e^3} \right] = \frac{4}{e^3} \approx .199$$

Probability Density Function

A real-valued function p such that $p(\vec{x}) \geq 0$ for all $\vec{x}$ and $\int_S p = 1$ where S is the set of all possibilities.

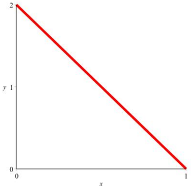
Example 1 Uniform Density: $p(x) = 1$ on [0,1]



$$\int_S p = \int_0^1 1 = x\Big|_0^1 = 1$$

Example 2: $p(x) = 2 - 2x$ on [0,1]

More likely to choose small numbers than larger numbers



Problem: Find the probability of picking a number less than 1/2.

$$\int_0^{1/2}(2-2x)\,dx = (2x - x^2)\Big|_0^{1/2} = (1 - \tfrac{1}{4}) - (0-0) = \tfrac{3}{4}$$

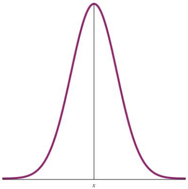
A probability density function on a set S in $\mathbb{R}^n$ is a continuous non-negative real-valued function $p : S \to \mathbb{R}^1$ such that

$$\int_S p dV = 1$$

If an experiment is performed where S is the set of all possible outcomes, then the probability that the outcome lies in a particular subset T is $\int_T p(\vec{x})\,dV$.

Example: **The Bell Curve**: The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$



Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Point of inflection at $(1, \frac{1}{\sqrt{e}}) = (1, .606)$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

Impossible to find antiderivative of $e^{-\frac{x^2}{2}}$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

Switch To Polar Coordinates: $A^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\, r\, d\theta\, dr$

$$A^2 = 2\pi \int_{r=0}^{\infty} re^{-\frac{r^2}{2}}\, dr = 2\pi \lim_{b\to\infty} \int_{r=0}^{b} re^{-\frac{r^2}{2}}\, dr$$

$$= 2\pi \lim_{b\to\infty} \left[-e^{-\frac{r^2}{2}}\right]_0^b = 2\pi \lim_{b\to\infty}\left[-\frac{1}{e^{b^2/2}} + \frac{1}{e^0}\right] = 2\pi \times 1 = 2\pi$$

Thus $A^2 = 2\pi$ so $A = \sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

To get a probability density, let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
This density is called the **Standard Normal Density**

Example: Suppose two numbers b and c are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing b and c is equivalent to choosing a point (b, c) from the unit square S with $p(\vec{x}) = 1$ (**Uniform Density**)

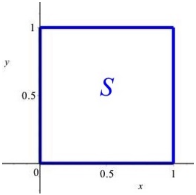
Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1.$

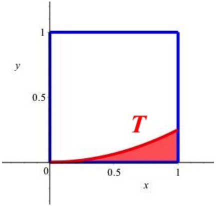
Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

$$\int_T p(\vec{x}) = \int_{x=0}^{1} \int_{y=0}^{x^2/4} 1 \, dy \, dx = \int_{x=0}^{1} \frac{x^2}{4} \, dx = \frac{x^3}{12}\bigg|_0^1 = \frac{1}{12}$$





General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \textstyle\int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

$$\text{Prob(2 lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

$$\text{More than } b \text{ hours: } e^{-3b\lambda}(1 + b\lambda)$$