MATH 223: Multivariable Calculus

Class 28: Wednesday April 23, 2025

Notes on Assignment 25

Assignment 26

Weighted Curves and Surfaces of Revolution

Announcements

Chapter 8: Integrals and Derivatives on Curves

Today: Weighted Curves and Surfaces of Revolution

Friday: Normal Vectors and Curvature

Monday: Flow Lines, Divergence and Curl

VECTOR FIELDS $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

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Definition The **Line Integral** of $\mathbf{F}$ over γ is

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = \int_a^b \mathbf{F}(g(t)) \cdot g'(t)\, dt$$

Alternative Notation for $n = 2$

$$g(T) = (g_1(t), g_2(t)) = (x(t), y(t))$$

$$\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$$

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = int_\gamma(F_1 dx + F_2 dy)$$

In our example, $\int_\gamma (x dx + y x^2 dy)$

Theorem The value of the line integral $\int_\gamma \mathbf{F}$ is independent of the parametrization of γ but in general is dependent on the curve itself.

For some vector fields, the line integral $\int_\gamma \mathbf{F}$ depends only on the **endpoints** of the curve.

Theorem (**The Fundamental Theorem of Calculus for Line Integrals**). Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable and let $\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Then $\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$.

If **F** $= \nabla f$ for some f, then we call **F**
a **Conservative Vector Field**
or an **Exact Vector Field**

and f is called a **Potential** of **F**

The function $P(\vec{x}) = -f(\vec{x})$ is the **Potential Energy** of the field **F**.

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$$\int_a^b mvv'\, dt = \frac{mv^2}{2}\Bigg|_{t=a}^{t=b}$$

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But $T(x)$= sum of Potential and Kinetic Energy

Law of Conservation of Total Energy

Arc Length

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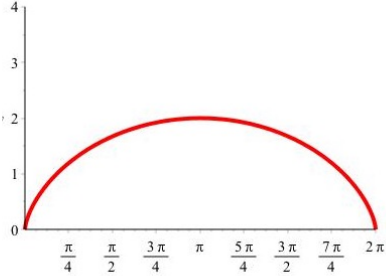

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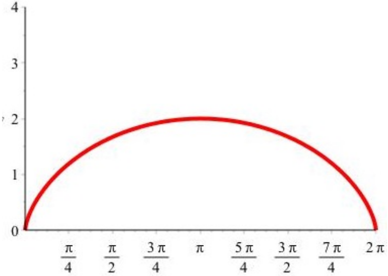

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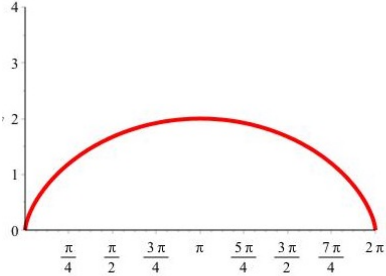

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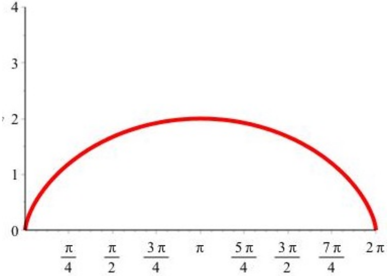

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$$L(\gamma) = \int_0^{2\pi} 2\sin(t/2)\, dt = -4\cos(t/2)\Big|_0^{2\pi} = 8$$

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If $g(t) = (h_1(t), h_2(t)|$, then $|g'(t)| = \sqrt{[h_1']^2 + [h_2']^2}$.

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Moving along the curve with uniform speed of 1 means that at time s we are at a point s units along the curve.

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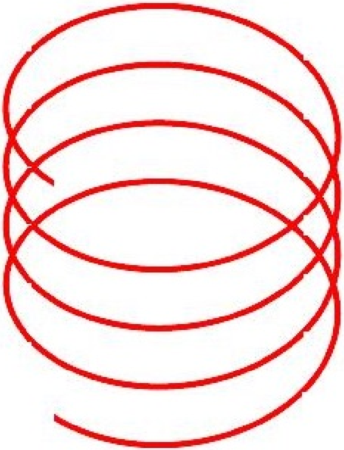

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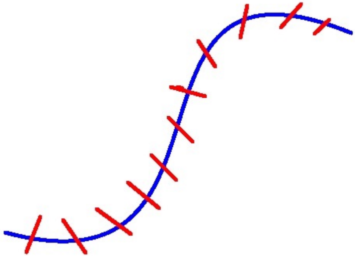

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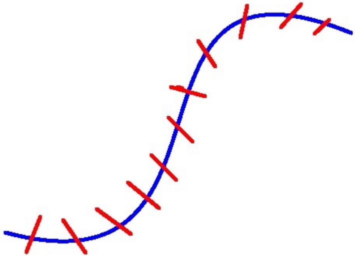

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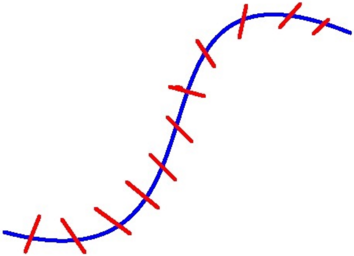

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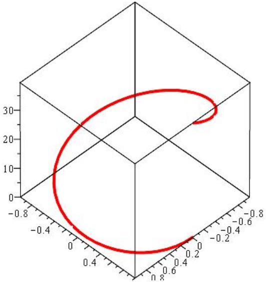

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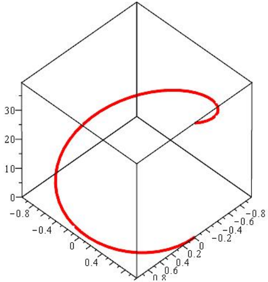

Suppose $\mu(x, y, z) = x^2 + y^2 + \sqrt{z} - 1$

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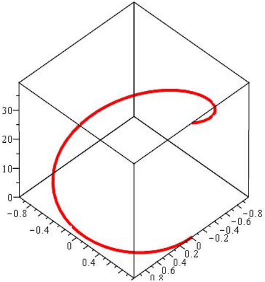

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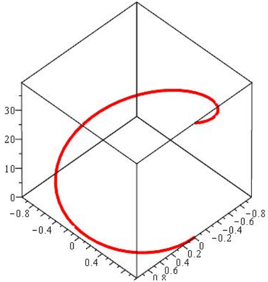

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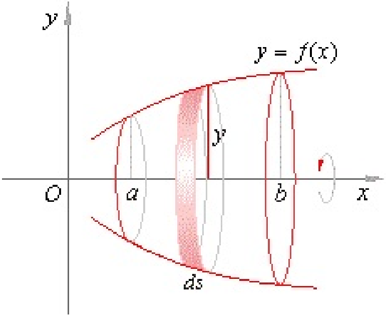
Thus Mass $= \int_0^{2\pi} t\sqrt{1 + 4t^2}\, dt$

$$= \frac{1}{12}(1 + 4t^2)^{3/2}\Big|_0^{2\pi} = \frac{1}{12}\left[(1 + 16\pi^2)^{3/2} - 1\right]$$

Surface of Revolution

S is a surface in $\mathbb{R}^3$ obtained by rotating a plane curve about a straight line in the plane.

Simplest Case: Rotate $y = f(x)$ about x-axis.



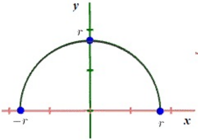
$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 \, dx$$

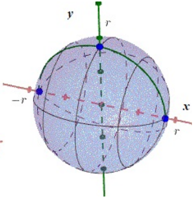
$$\text{Surface Area} = \int_a^b 2\pi \sqrt{1 + \left[f(x)\right]^2} \, dx$$

$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 \, dx$$
$$\text{Surface Area} = \int_a^b 2\pi \sqrt{1 + \left[f(x)\right]^2} \, dx$$
Suppose curve has parametrization $g : \mathbb{R}^1 \to \mathbb{R}^2, t_0 \leq t \leq t_1$

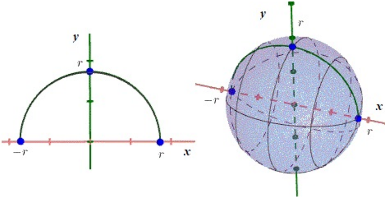
$g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.
$$\text{Volume} = \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t) \, dt$$
$$\text{Surface Area} = \int_{t_0}^{t_1} 2\pi y(t) |g'(t)| \, dt$$

Example Revolve Semicircle of radius r about horizontal axis.



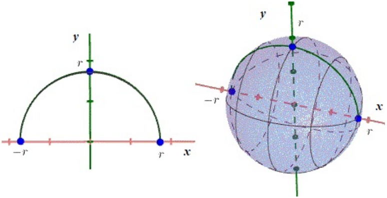


Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r\cos t, r\sin t), 0 \leq t \leq \pi$$

Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r\cos t, r\sin t), 0 \le t \le \pi$$
$$\text{Volume} = \int_{t_0}^{t_1} \pi \, [y(t)]^2 \, x'(t) \, dt$$
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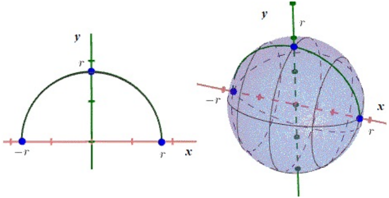

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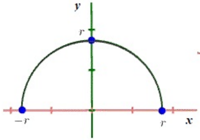
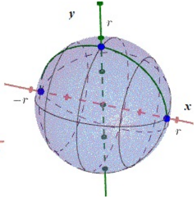
Example Revolve Semicircle of radius r about horizontal axis.





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$$\text{Surface Area } = \int_{t_0}^{\pi} r^2 2\pi \sin t\, dt$$
$$= -2\pi r^2 \cos t\Big|_0^{\pi} = -2r^2\pi(-1-1) = 4\pi r^2.$$

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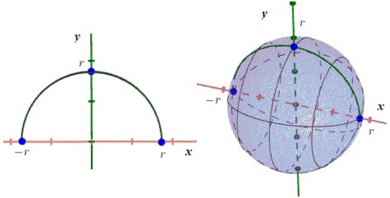

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$$= -2\pi r^2 \cos t\Big|_0^\pi = -2r^2\pi(-1-1) = 4\pi r^2.$$
$$\text{Volume} = \int_0^\pi \pi(r\sin t)^2 r\sin t\, dt = \tfrac{4}{3}\pi r^3$$