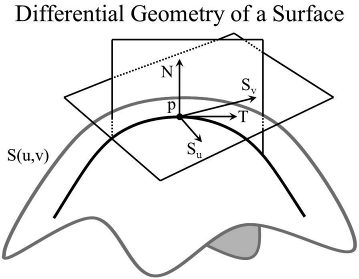
MATH 223: Multivariable Calculus



Class 29: April 23, 2025

Notes on Assignment 27

Assignment 28

Normal Vectors and Curvature

"Sit and stay were no problem but she's hit a wall with multivariable calculus."

Exam 3: Wednesday Night at 7 PM

You May Bring One Sheet (Two-Sided) of Notes

Announcements

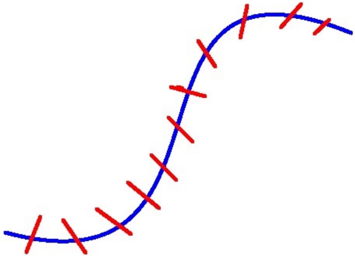
Chapter 7: Integrals and Derivatives on Curves

Today: Weighted Curves and Surfaces of Revolution
Conservation of Energy
Normal Vectors and Curvature

After Thanksgiving: Monday: Flow Lines, Divergence and Curl
Wednesday: Conservative Vector Fields

Mass of a Weighted Curve

Density (μ) is mass per unit length



Total Mass $\sim \sum \mu(point)\times$ Length of short piece of curve

Total **Mass** $= \int \mu(g(t))|g'(t)|\,dt$

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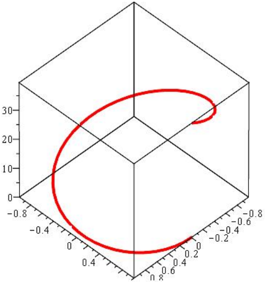

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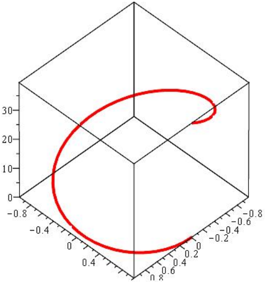

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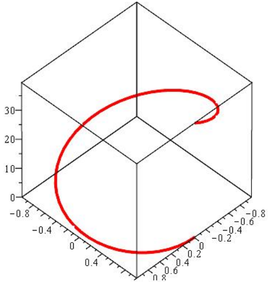

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Then $\mu(g(t)) = \mu(\sin t, \cos t, t^2) = \cos^2 + \sin^2 t + \sqrt{t^2} - 1$

$= 1 + t - 1 = t$

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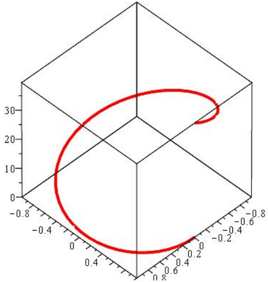

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Thus **Mass** $= \int_0^{2\pi} t\sqrt{1 + 4t^2}\,dt$

$$= \frac{1}{12}(1 + 4t^2)^{3/2}\Big|_0^{2\pi} = \frac{1}{12}\left[(1 + 16\pi^2)^{3/2} - 1\right]$$

Surface of Revolution

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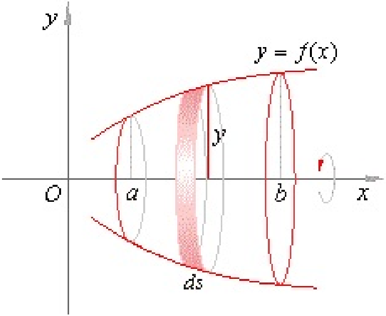
Simplest Case: Rotate $y = f(x)$ about x-axis.



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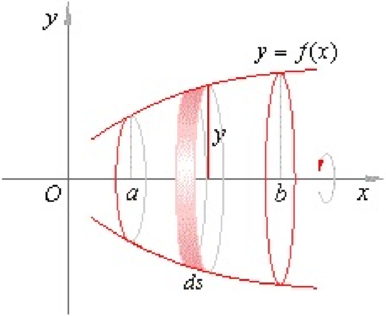

$$\text{Volume } = \int_a^b \pi \left[f(x)\right]^2 \; dx$$

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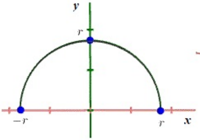
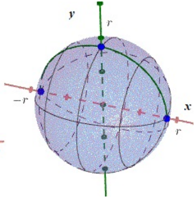
$g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.

Volume $= \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t) \, dt$

Surface Area $= \int_{t_0}^{t_1} 2\pi y(t)|g'(t)| \, dt$

Example Revolve Semicircle of radius r about horizontal axis.





$$g(t) = (r\cos t, r\sin t), 0 \le t \le \pi$$

$$\text{Volume} = \int_{t_0}^{t_1} \pi\,[y(t)]^2\, x'(t)\, dt$$

$$\text{Surface Area} = \int_{t_0}^{t_1} 2\pi y(t)|g'(t)|\, dt$$

$$\text{Surface Area} = \int_{t_0}^{\pi} r^2 2\pi \sin t\, dt$$

$$= -2\pi r^2 \cos t\Big|_0^{\pi} = -2r^2\pi(-1-1) = 4\pi r^2.$$

$$\text{Volume} = \int_0^{\pi} \pi (r\sin t)^2 r \sin t\, dt = \frac{4}{3}\pi r^3$$

If **F** $= \nabla f$ for some f, then we call **F**
a **Conservative Vector Field**
or an **Exact Vector Field**

and f is called a **Potential** of **F**

The function $P(\vec{x}) = -f(\vec{x})$ is the **Potential Energy** of the field **F**.

Conservative Vector Field: $\mathbf{F}(x, y) = (2xy, x^2 + 2y)$
Nonconservative Example $\mathbf{F}(x, y) = (x, x + 1)$

Application: **Conservation of Energy**

Suppose $\mathbf{g}(t)$ represents the position of an object of varying mass $m(t)$ in space at time t.

The velocity vector of the object is $\mathbf{v} = \mathbf{g}'(t)$.

The Force acting on the object at position $g(t)$ is

$$\mathbf{F}(\mathbf{g}(t)) = [m(t)\mathbf{v}(t)]' = m'(t)\mathbf{v}(t) + m(t)\mathbf{v}'(t)$$

Then

$$\begin{aligned}\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) &= \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{v}(t)\\ &= \left[m'(t)\mathbf{v}(t) + m(t)\mathbf{v}'(t)\right] \cdot \mathbf{v}(t)\\ &= m'(t)\mathbf{v}(t) \cdot \mathbf{v}(t) + m(t)\mathbf{v}'(t) \cdot \mathbf{v}(t)\\ &= m'(t)s^2(t) + m(t)s'(t)s(t)\end{aligned}$$

where $s(t) = |\mathbf{v}(t)| =$ **speed** at time t.

To Show: $s'(t)s(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t)$

Start with $s^2(t) = |\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$

Differentiate each side with respect to t:

$$2s(t)s'(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}'(t) \cdot \mathbf{v}(t)$$

Thus $s'(t)s(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t)$
and

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)s^2(t) + m(t)s'(t)s(t)$$

Application: Conservation of Energy

(a) $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)s^2(t) + m(t)s'(t)s(t)$

We'll use the scalar v for the scalar s

so $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)v^2(t) + m(t)v'(t)v(t)$

(b) $m(t) =$ Constant implies $m' = 0$

so $\mathbf{F}(g(t)) \cdot g'(t) = mv(t)v'(t)$

$$\int_a^b mv(t)v'(t)\,dt = \frac{mv(t)^2}{2}\Bigg|_{t=a}^{t=b}$$

Application: **Conservation of Energy**

Suppose **F** is a force field which moves an object of mass m from $\vec{a}$ to $\vec{b}$ along curve γ.

Let g be a parametrization of curve γ and $v(t) = g'(t)$.

Then the work done in moving the object is

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2$$ (**Change in Kinetic Energy)**

If **F** is a conservative field, then we can also compute work done by $\int_\gamma \mathbf{F} = f(\vec{b}) - f(\vec{a}) = p(\vec{a}) - p(\vec{b}) =$ **Change in Potential Energy**

Equating the two expressions for work, we have

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2 = p(\vec{a}) - p(\vec{b})$$

$$p(\vec{b}) + \frac{1}{2}m|v(t_b)|^2 = p(\vec{a}) + \frac{1}{2}m|v(t_a)|^2$$

where $\vec{a}$ and $\vec{b}$ are any 2 points

So Sum of Potential and Kinetic Energy is Constant

Law of Conservation of Total Energy

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Goal: Derive a Measure of Shape of a Curve.

How "Curvy" is a Curve?

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Parametrization $\mathbf{g}$ whose image is γ.

Some texts use $\mathbf{r}$ or $\mathbf{x} = \mathbf{x}(t)$ for the parametrization

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Arc Length traversed by time t is denoted $s(t)$ and is a scalar quantity with

$$s(t) = \int |\mathbf{g}'(t)| \, dt$$

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Speed is Derivative of Arc Length:

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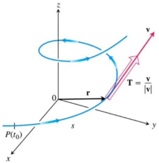
so we will have $\mathbf{g}'(t) = s'(t)\mathbf{T}(t)$

where **T** is unit tangent vector $\frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$

Unit Tangent Vector

The **unit tangent vector** gets its own notation:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\left|\vec{\mathbf{r}}\,'(t)\right|} = \frac{\vec{\mathbf{v}}}{\left|\vec{\mathbf{v}}\right|}$$



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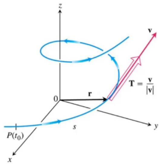

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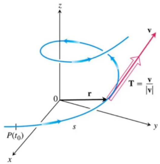

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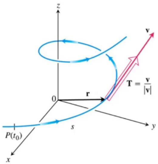

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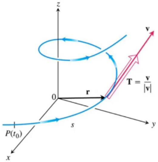

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Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

Principal Normal Vector

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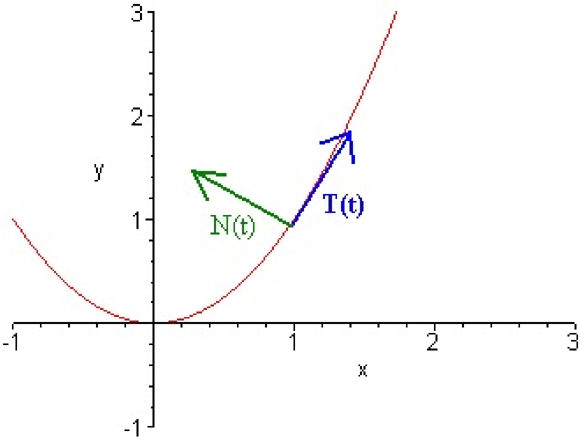
The Principal Normal Vector

$$\eta(t) = \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

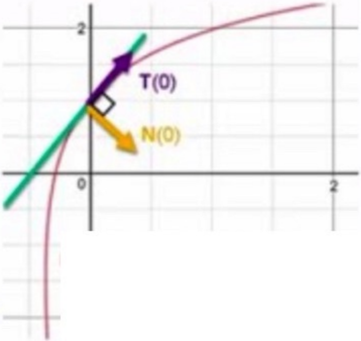
Sometimes written as $\mathbf{N} = \frac{\mathbf{T'}}{|\mathbf{T'}|}$ or $\mathbf{n} = \frac{\mathbf{t}}{|\mathbf{t}|}$



3
2
1
-1
y
N(t)
T(t)
-1
1
2
3
x

Principal Unit Normal Vector

- $0 = 1' = (T \cdot T)'$
  $= T' \cdot T + T \cdot T'$
  $= 2T \cdot T'$



- $$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

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$$\mathbf{N} = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}} \times \frac{\sqrt{a^2+b^2}}{a} = \frac{(-a\cos t, -a\sin t, 0)}{a}$$

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Check that $\mathbf{N} \cdot \mathbf{T} = 0$

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Recall $s'(t) = |\mathbf{g}'(t)|$ or, more compactly, $s' = |\mathbf{g}'|$ and $\mathbf{T} = \frac{\mathbf{g}'}{|\mathbf{g}'|} = \frac{\mathbf{g}'}{s'}$ we have $\mathbf{g}' = s'\mathbf{T}$.

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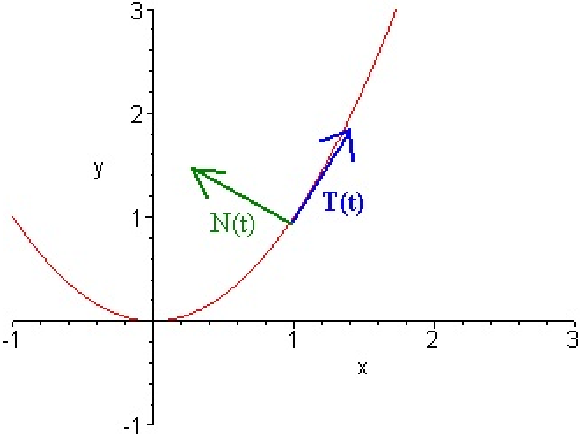

3
2
1
-1
y
-1
1
2
3
x
N(t)
T(t)

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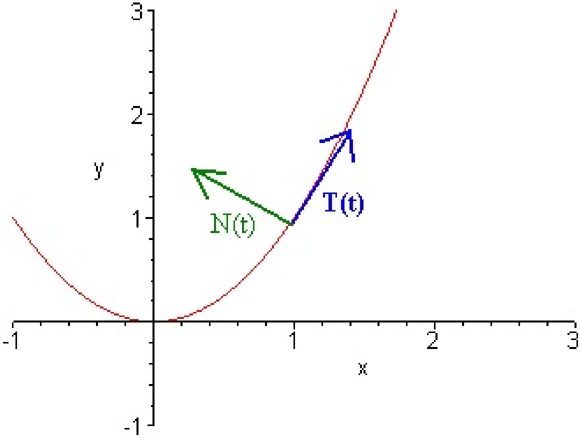
Thus Curvature $= \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|} = \frac{2}{1+4t^2} \times \frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}$

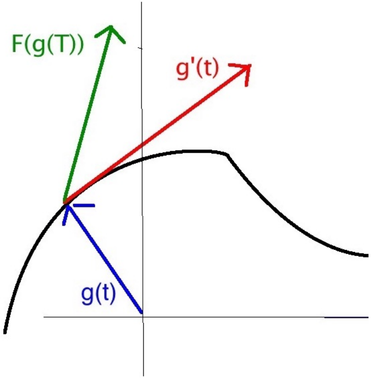


Flow Lines

Suppose γ is a curve in $\mathbb{R}^n$ which has a parametrization g.
At each point on the curve, we can associate two vectors:

Tangent Vector: $\mathbf{g'}(t)$

Vector Field: $\mathbf{F}(\mathbf{g}(t))$



If the two vectors coincide, then γ is called a **flow line** for **F**.

Hard Problem: Given $\mathbf{F}$, find flow lines
(Central Question in Differential Equations)

Easy Problem: Given $\mathbf{g}$ and $\mathbf{F}$, check if γ is a flow line for $\mathbf{F}$.

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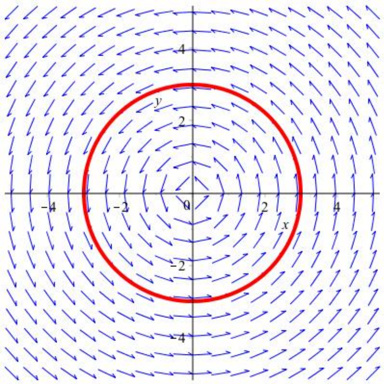

Flow Lines and Differential Equations

Star with a system of differential equations

$$\frac{dx}{dt} = (2 - y)(x - y) = f(x, y)$$

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Definition:A **flow line** of a vector field **F** is a differentiable function **g** such that the velocity vector **g'** at each point coincides with the field vector **F(g)**.