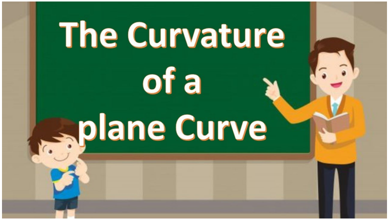
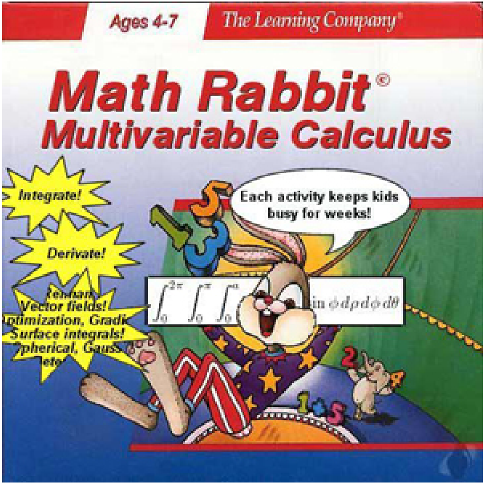
MATH 223: Multivariable Calculus



Class 30: Monday, April 28, 2025

Assignment 27 (Due Wednesday)

Normal Vectors and Curvature



Exam 3: Tonight at 7 PM

You May Bring One Sheet (Two-Sided) of Notes

Announcements

Chapter 7: Integrals and Derivatives on Curves

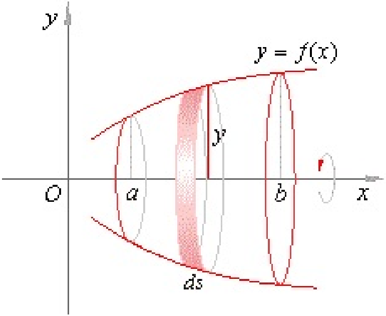
Today: Weighted Curves and Surfaces of Revolution
Conservation of Energy
Normal Vectors and Curvature

Monday: Flow Lines, Divergence and Curl
Wednesday: Conservative Vector Fields

Surface of Revolution

S is a surface in $\mathbb{R}^3$ obtained by rotating a plane curve about a straight line in the plane.

Simplest Case: Rotate $y = f(x)$ about x-axis.



$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 \, dx$$

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx$$

Volume $= \int_a^b \pi \left[f(x)\right]^2 \, dx$

Surface Area $= \int_a^b 2\pi f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx$

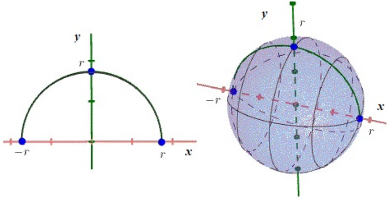
Suppose curve has parametrization $g : \mathbb{R}^1 \to \mathbb{R}^2, t_0 \leq t \leq t_1$

$g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.

Volume $= \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t)\, dt$

Surface Area $= \int_{t_0}^{t_1} 2\pi y(t)|g'(t)|\, dt$

Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r\cos t, r\sin t), 0 \le t \le \pi$$

$$\text{Volume } = \int_{t_0}^{t_1} \pi\,[y(t)]^2\, x'(t)\, dt$$

$$\text{Surface Area } = \int_{t_0}^{t_1} 2\pi y(t)|g'(t)|\, dt$$

$$\text{Surface Area } = \int_{t_0}^{\pi} r^2 2\pi \sin t\, dt$$

$$= -2\pi r^2 \cos t\Big|_0^{\pi} = -2r^2\pi(-1-1) = 4\pi r^2.$$

$$\text{Volume } = \int_0^{\pi} \pi(r\sin t)^2 r\sin t\, dt = \tfrac{4}{3}\pi r^3$$

Normal Vectors and Curvature

Goal: Derive a Measure of Shape of a Curve.
How "Curvy" is a Curve?
Setting: Curve γ lies in $\mathbb{R}^2$ or $\mathbb{R}^3$
Parametrization $\mathbf{g}$ whose image is γ.
Some texts use $\mathbf{r}$ or $\mathbf{x} = \mathbf{x}(t)$ for the parametrization
Arc Length traversed by time t is denoted $s(t)$ and is a scalar quantity with

$$s(t) = \int |\mathbf{g}'(t)|\, dt$$

Arc Length is Integral of Speed
Speed is Derivative of Arc Length:

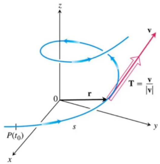
$$s'(t) = |\mathbf{g}'(t)|$$

so we will have $\mathbf{g}'(t) = s'(t)\mathbf{T}(t)$
where $\mathbf{T}$ is unit tangent vector $\frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$

Unit Tangent Vector

The **unit tangent vector** gets its own notation:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{|\vec{\mathbf{r}}\,'(t)|} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$



$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$$

<u>Example</u> $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2 + b^2}$

$$\mathbf{T}(t) = \frac{g'(t)}{|g'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

Principal Normal Vector

Start With Observation: $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$

Now differentiate both sides with respect to t:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 2\mathbf{T} \cdot \mathbf{T}' = 0$$

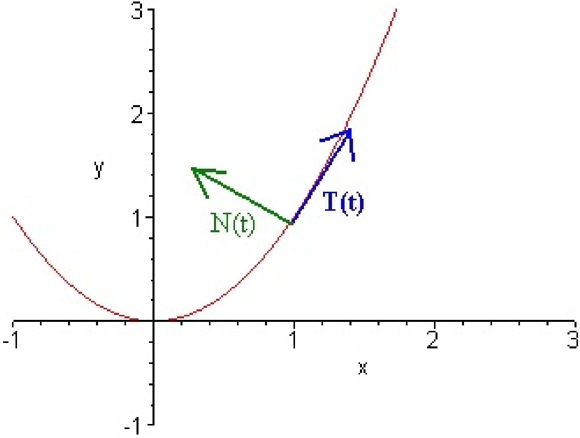
So $\mathbf{T} \cdot \mathbf{T}' = 0$

The vectors $\mathbf{T}$ and $\mathbf{T}'$ are Orthogonal

The Principal Normal Vector

$$\eta(t) = \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

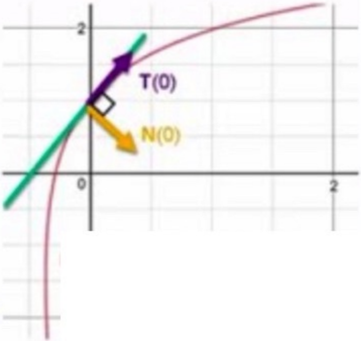
Sometimes written as $\mathbf{N} = \frac{\mathbf{T'}}{|\mathbf{T'}|}$ or $\mathbf{n} = \frac{\mathbf{t}}{|\mathbf{t}|}$



3
2
1
-1
y
x
-1
1
2
3
N(t)
T(t)

Principal Unit Normal Vector

- $0 = 1' = (T \cdot T)'$
  $= T' \cdot T + T \cdot T'$
  $= 2T \cdot T'$



- $$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Principal Normal

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

<u>Example</u> $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2 + b^2}$

$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

$$\mathbf{N} = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}} \times \frac{\sqrt{a^2+b^2}}{a} = \frac{(-a\cos t, -a\sin t, 0)}{a}$$

$$\mathbf{N} = (-\cos t, -\sin t, 0)$$

$$\mathbf{N}\cdot\mathbf{T} = \frac{a\sin t\cos t - a\sin t\cos t + 0}{\sqrt{a^2+b^2}} = 0.$$

Example: Parabola in the Plane

$$\mathbf{g}(t) = (t, t^2)$$

$$\mathbf{g}'(t) = (1, 2t)$$

$$|\mathbf{g}'(t)| = \sqrt{1 + 4t^2}$$

$$\mathbf{T} = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(1,2t)}{\sqrt{1+4t^2}} = \left((1+4t^2)^{-1/2}, 2t(1+4t^2)^{-1/2}\right)$$

Differentiating with respect to t and simplifying, we get

$$\mathbf{T}' = \left(\frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$$

After some algebra, $|\mathbf{T}'| = \frac{2}{1+4t^2}$

$$\mathbf{N} = \left(\frac{-2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}},\right)$$

Check that $\mathbf{N} \cdot \mathbf{T} = 0$

Curvature

Recall $s'(t) = |\mathbf{g}'(t)|$ or, more compactly, $s' = |\mathbf{g}'|$ and $\mathbf{T} = \frac{\mathbf{g}'}{|\mathbf{g}'|} = \frac{\mathbf{g}'}{s'}$ we have $\mathbf{g}' = s'\mathbf{T}$.

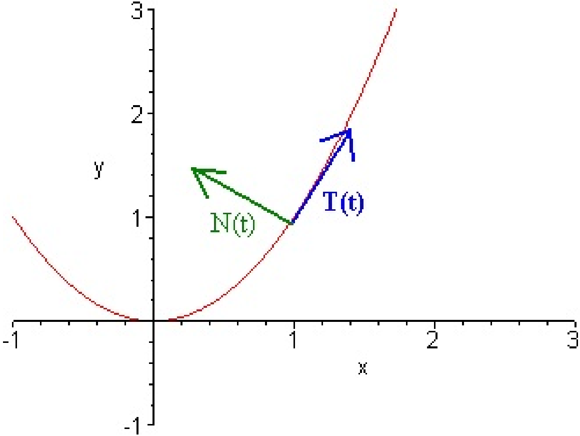
Differentiate with respect to t:

$$\mathbf{g}'' = \mathbf{g}'' = (s'\mathbf{T})' = s''\mathbf{T} + s'\mathbf{T}'$$

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\mathbf{T}'$
acceleration vector		component in direction of $\mathbf{T}$		component in direction of $\mathbf{T}'$

Replace $\mathbf{T}'$ by $|\mathbf{T}'|\mathbf{N}$:

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'|\mathbf{T}'|\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration


3
2
1
-1
y
x
-1
1
2
3
N(t)
T(t)

Curvature

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\lvert\mathbf{T}'\rvert\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration

Curvature is a measure of the bend

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right|$$

Theorem: $\kappa = \frac{|\mathbf{T}'|}{s'} = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}$.

Proof: $\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\mathbf{T}'}{s'}$

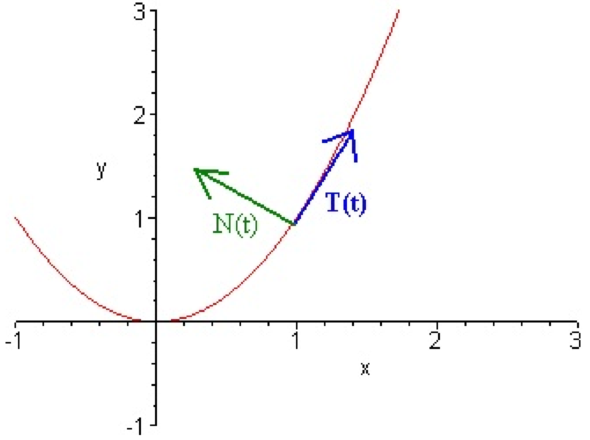
$$\boxed{\kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}}$$

$$\boxed{\text{Curvature: } \kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}}$$

Example Our Parabola $\mathbf{g}(t) = (t, t^2)$

We found $|\mathbf{T}'| = \frac{2}{1+4t^2}$ and $|\mathbf{g}'(t)| = \sqrt{1+4t^2}$

Thus Curvature $= \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|} = \frac{2}{1+4t^2} \times \frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}$

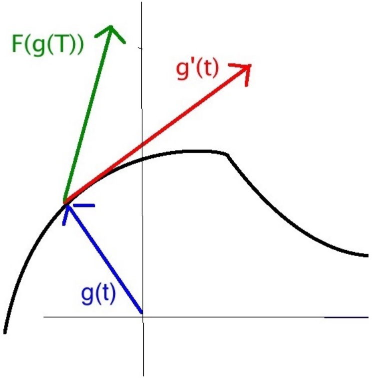


Flow Lines

Suppose γ is a curve in $\mathbb{R}^n$ which has a parametrization g.
At each point on the curve, we can associate two vectors:

Tangent Vector: $\mathbf{g'}(t)$

Vector Field: $\mathbf{F}(\mathbf{g}(t))$



If the two vectors coincide, then γ is called a **flow line** for $\mathbf{F}$.

Hard Problem: Given $\mathbf{F}$, find flow lines
(Central Question in Differential Equations)

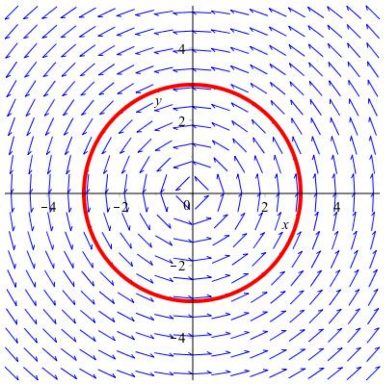
Easy Problem: Given $\mathbf{g}$ and $\mathbf{F}$, check if γ is a flow line for $\mathbf{F}$.

Example: $\mathbf{g}(t) = (3\cos\frac{t}{12}, 3\sin\frac{t}{12})$

Then $\mathbf{g'}(t) = (-\frac{1}{4}\sin\frac{t}{12}, \frac{1}{4}\cos\frac{t}{12})$

Suppose $\mathbf{F}(x,y) = \left(\frac{-y}{4\sqrt{x^2+y^2}}, \frac{x}{4\sqrt{x^2+y^2}}\right)$

Then $\mathbf{F}(x,y) = \left(\frac{-3\sin\frac{t}{12}}{4\times 3}, \frac{3\cos\frac{t}{12}}{4\times 3}\right) = \mathbf{g'}(t)$



Flow Lines and Differential Equations

Star with a system of differential equations

$$\frac{dx}{dt} = (2-y)(x-y) = f(x,y)$$
$$\frac{dy}{dt} = (1+x)(x+y) = g(x,y)$$

Can write as a single equation:

$$\frac{dy}{dx} = \frac{(1+x)(x-y)}{(2-y)(x-y)} = \frac{g(x,y)}{f(x,y)}$$

Observe:

1. Solution of the equation is a curve in the (x, y)-plane
2. As time goes forward, point moves along the curve in accordance to the equation
3. $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ is a vector field.
4. At each point on curve, direction of motion is given by the vector field
5. The vector field is tangent to the curve
6. The curve is tangent to the vector field

Definition: A **flow line** of a vector field **F** is a differentiable function **g** such that the velocity vector **g'** at each point coincides with the field vector **F(g)**.