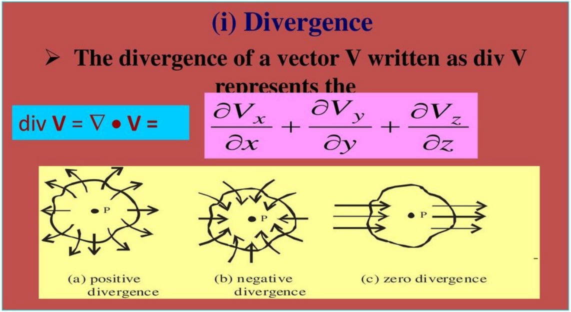
MATH 223: Multivariable Calculus

Divergence, Curl and Gradient Operations



Class 31: Wednesday, April 30, 2025

Handouts

Notes on Exam 3

Notes on Assignment 27

Assignment 28

Divergence and Curl

Today

Flow Lines

Begin Chapter 8: Vector Field Theory

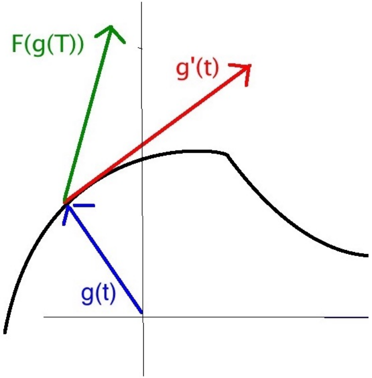
Divergence and Curl: Measures of Rates of Change of Vector Fields

Flow Lines

Suppose γ is a curve in $\mathbb{R}^n$ which has a parametrization g.
At each point on the curve, we can associate two vectors:

Tangent Vector: $\mathbf{g'}(t)$

Vector Field: $\mathbf{F}(\mathbf{g}(t))$



If the two vectors coincide, then γ is called a **flow line** for **F**.

Hard Problem: Given $\mathbf{F}$, find flow lines
(Central Question in Differential Equations)

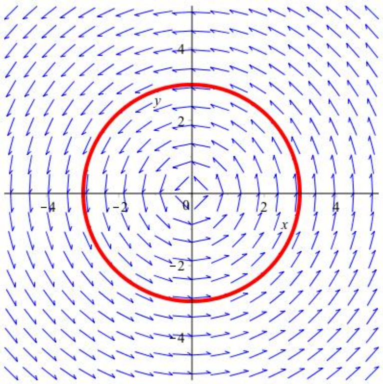
Easy Problem: Given $\mathbf{g}$ and $\mathbf{F}$, check if γ is a flow line for $\mathbf{F}$.

Example: $\mathbf{g}(\mathrm{t}) = (3\cos\frac{t}{12}, 3\sin\frac{t}{12})$

Then $\mathbf{g'}(\mathrm{t}) = (-\frac{1}{4}\sin\frac{t}{12}, \frac{1}{4}\cos\frac{t}{12})$

Suppose $\mathbf{F}(x,y) = \left(\frac{-y}{4\sqrt{x^2+y^2}}, \frac{x}{4\sqrt{x^2+y^2}}\right)$

Then $\mathbf{F}(x,y) = \left(\frac{-3\sin\frac{t}{12}}{4\times 3}, \frac{3\cos\frac{t}{12}}{4\times 3}\right) = \mathbf{g'}(t)$



Flow Lines and Differential Equations

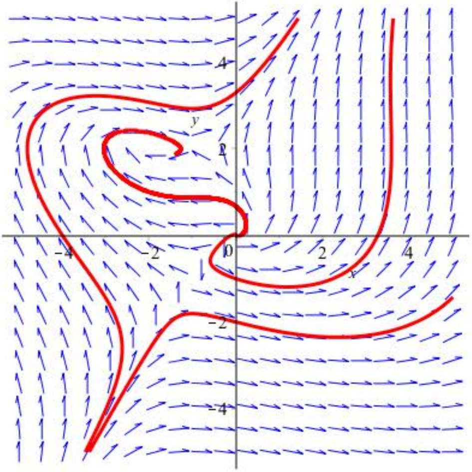
Star with a system of differential equations

$$\frac{dx}{dt} = (2-y)(x-y) = f(x,y)$$
$$\frac{dy}{dt} = (1+x)(x+y) = g(x,y)$$

Can write as a single equation:
$$\frac{dy}{dx} = \frac{(1+x)(x-y)}{(2-y)(x-y)} = \frac{g(x,y)}{f(x,y)}$$
Observe:

1. Solution of the equation is a curve in the (x, y)-plane
2. As time goes forward, point moves along the curve in accordance to the equation
3. $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ is a vector field.
4. At each point on curve, direction of motion is given by the vector field
5. The vector field is tangent to the curve
6. The curve is tangent to the vector field

Definition: A **flow line** of a vector field **F** is a differentiable function **g** such that the velocity vector **g'** at each point coincides with the field vector **F(g)**.



Divergence of a Vector Field

Definition div $\mathbf{F}$ = trace of $\mathbf{F}'$ of the Jacobi Matrix

Example **F**: $\mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (2x - y, x - 3y)$

$$\mathbf{F} = \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} \text{ implies div } \mathbf{F} = 2 - 3 = -1$$

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x, y, z) = (xy, yz, zx)$

$$\mathbf{F}' = \begin{pmatrix} y & -- & -- \\ -- & z & -- \\ -- & -- & x \end{pmatrix} \text{ implies div } \mathbf{F} = y + z + x$$

Example: $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x, y, z) = (yz, xz, xy)$

Alternate Notation: $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$\mathbf{F'} = \begin{pmatrix} 0 & z & y \\ z & 0 & y \\ y & z & 0 \end{pmatrix} \text{ implies div } \mathbf{F} = 0$$

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

$$\mathbf{F'} = \begin{matrix} F1 \\ F2 \\ F3 \end{matrix} \begin{pmatrix} 0 & z & y \\ z & 0 & y \\ y & z & 0 \end{pmatrix} \text{ implies div } \mathbf{F} = 0$$

In general, **div F is a real -valued function of n variables**.

Notes

1. Gauss's Theorem: $\int_R \text{ div } \mathbf{F} \, dV = \int_{\partial R} \mathbf{F} \cdot d\mathbf{S}$

2. div $\mathbf{F}$ gives expansion rate of fluid at point $\mathbf{x}$
   div $\mathbf{F} > 0$ means fluid is expanding, getting less dense
   div $\mathbf{F} < 0$ means fluid is contracting, becomes more dense

3. Alternate Notation; $\mathbf{F} = (F_1, F_2, F_3), \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
   Then div $\mathbf{F} = \mathbf{F} \cdot \nabla$

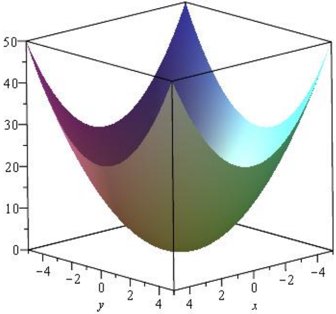
Example

$$\mathbf{F}(x,y,z) = (xy^2 + z\ln(1+y^2), \sin(xz) - zy, x^2z + \arctan y + e^{x^2})$$

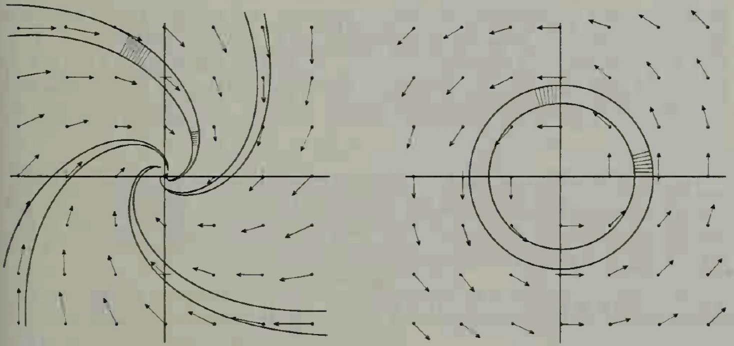
$$\text{div } \mathbf{F} = y^2 - z + x^2$$

so div $\mathbf{F} > 0$ if $x^2 + y^2 > z$

$z = x^2 + y^2$ is equation of elliptic paraboloid.



Divergence is positive on the outside, negative on the inside.



$\mathbf{F}(x, y) = \frac{1}{8}(-x + y)\mathbf{i} + \frac{1}{8}(-x - y)\mathbf{j}$
Area decreased: $\text{div}\mathbf{F}(x, y) = -\frac{1}{4}$

(a)

$\mathbf{G}(x, y) = -\frac{1}{4}(y/\sqrt{x^2 + y^2})\mathbf{i} + \frac{1}{4}(x/\sqrt{x^2 + y^2})\mathbf{j}$
Area preserved: $\text{div}\mathbf{G}(x, y) = 0$

(b)

Figure 8.10

Curl of a Vector Field

Curl measures local tendency of a vector field and its flow lines to circulate around some axis.

The curl of a vector field is itself a vector field.

Setting; $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is our vector field

$\mathbf{F} = (F_1, F_2, F_3)$ so $\mathbf{F}(x, yz) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

Formal Definition: curl $\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

Mnemonic Device:

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Expand along first row:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$$

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example: $\mathbf{F}(x, y, z) = (xyz, y - 3z, 2y)$

$$\text{curl } \mathbf{F} = ((2y)_y - (-3z)_z, (xyz)_z - (2y)_x, (y - 3z)_x - (xyz)_y)$$

$$= (2 - (-3), xy - 0, 0 - xz) = (5, xy, -xz)$$

Scalar Curl for Vector Fields in Plane

F $= (F, G, 0)$ where$F(x, y)$ and $G(x, y)$are functions only of x and y.

Then curl **F** $= (0, 0, G_x - F_y)$

Note: Curl and Conservative Vector Field

Suppose **F** $= (F, G, 0)$ is gradient field with **F** $= \nabla f$.

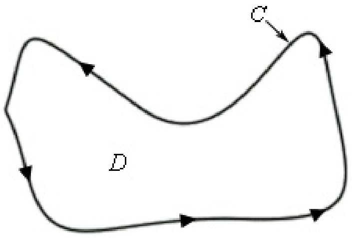
Then $F = f_x$ and $G = f_y$

In this case, Curl **F** $= (0, 0, f_{yx} - f_{xy}) = (0, 0, 0)$

by Clairault's Theorem on Equality of Mixed Partials.

Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$



D is bounded plane region.
$C = \gamma$ is piecewise smooth boundary of D
F and G are continuously differentiable functions defined on D
Then

$$\int \int (G_x - F_y) dx dy = \int_\gamma F dx + G y$$

where γ is parametrized so it is traced once with D on the left.

Application of Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$

Example $\mathbf{F}(x,y) = (0,x)$ implies $curl\mathbf{F} = 1 - 0 = 1$

Hence $\iint_D \text{curl } \mathbf{F} = \iint_D 1 =$ area of D

Green's Theorem enables us to find the area of a planar region if we can develop a parametrization of its boundary.

Example Consider the unit disk D of radius r centered at origin.

Let $g(t) = (r\cos t, r\sin t), 0 \leq t \leq 2\pi$

So $g'(t) = (r\sin t, r\cos t)$

and $\mathbf{F}(g(t)) = (0, r\cos t)$

Then $\mathbf{F}(g(t)) \cdot g'(t) = r^2\cos^2 t\, dt$

Thus area of $D = \iint_D 1 = \iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F} = \int_0^{2\pi} r^2\cos^2 t\, dt$

$$\int_0^{2\pi} r^2\cos^2 t\, dt = r^2\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = \frac{r^2}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi} \pi r^2$$

Using Green's Theorem

(1) Compute $\iint_D \operatorname{curl} \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \operatorname{curl} \mathbf{F}$

Curl of a Vector Field

Curl measures local tendency of a vector field and its flow lines to circulate around some axis.

The curl of a vector field is itself a vector field.

Setting; $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is our vector field

$\mathbf{F} = (F_1, F_2, F_3)$ so $\mathbf{F}(x, yz) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

Formal Definition: curl $\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

Mnemonic Device:

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Expand along first row:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$$

Scalar Curl for Vector Fields in Plane

$\mathbf{F} = (F, G, 0)$ where$F(x, y)$ and $G(x, y)$are functions only of x and y.

Then curl $\mathbf{F} = (0, 0, G_x - F_y)$

Note: Curl and Conservative Vector Field

Suppose $\mathbf{F} = (F, G, 0)$ is gradient field with $\mathbf{F} = \nabla f$.

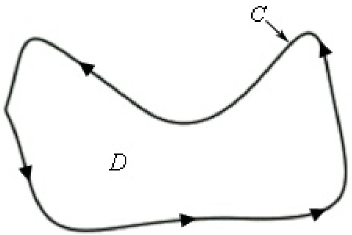
Then $F = f_x$ and $G = f_y$

In this case, Curl $\mathbf{F} = (0, 0, f_{yx} - f_{xy}) = (0, 0, 0)$

by Clairault's Theorem on Equality of Mixed Partials.

Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$



D is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of D

F and G are continuously differentiable functions defined on D

Then

$$\int\int (G_x - F_y) dx dy = \int_\gamma F dx + G y$$

where γ is parametrized so it is traced once with D on the left.

Using Green's Theorem

(1) Compute $\iint_D \operatorname{curl} \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

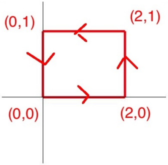
(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \operatorname{curl} \mathbf{F}$

Example

Find

$$\int_\gamma (1+10xy+y^2)\,dx+(6xy+5x^2)\,dy = \int_\gamma (1+10xy+y^2, 6xy+5x^2)$$

where γ is boundary of the rectangle with vertices (0,0), (2,0), (2,1), and (0,1).



Note: Direct Computation requires 4 integrals.

$$F(x,y) = 1 + 10xy + y^2. \quad G(x,y) = 6xy + 5x^2$$

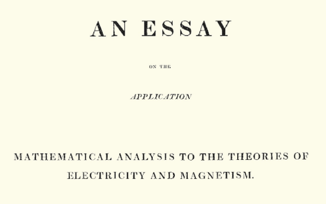
$$F_y = 10x + 2y \qquad . \; G_x = 6y + 10x$$

$$G_x - F_y = 6y + 10x - 10x - 2y = 4t$$

$$\int_\gamma \mathbf{F} = \iint_D \text{curl } \mathbf{F} = \int_0^2 \int_0^1 4y\,dy\,dx = \int_0^2 \left[2y^2\right]_0^1 = \int_0^2 2dx = 4$$

George Green
1793 – 1841

Mikhail Ostrogradsky
1801 – 1861



AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.