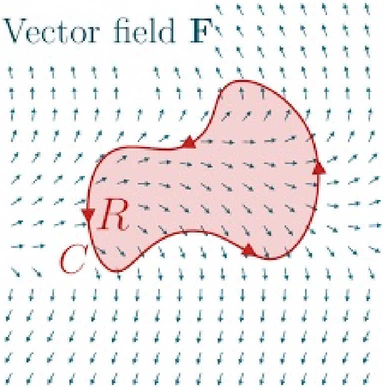
# MATH 223: Multivariable Calculus




Class 32: Friday, May 2025

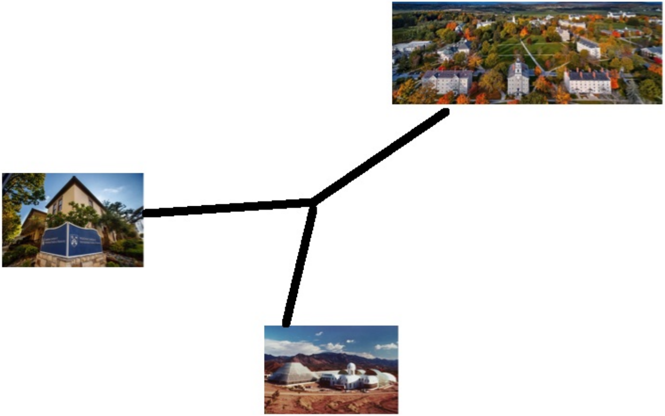
# Notes on Assignment 28
# Assignment 29
# Green'sTheorem
# Using MATLAB to Locate and Identify Extreme Points

## Rough Weights for Course Components

| | |
|---|---|
| Exam1 | 20% |
| Exam 2 | 20% |
| Exam 3 | 20% |
| Final Exam | 30% |
| Project | 10 % |

**Final Exam**
**Thursday, May 15**
**7 PM – 10 PM**

# Location Problem



Due: Friday, May 9

Announcements

**Today**

**More Green's Theorem**

**Conservative Vector Fields**

## Divergence of a Vector Field

Definition div **F** $=$ trace of **F'**, the Jacobi Matrix

In general, div **F** is a real -valued function of $n$ variables.

# Curl of a Vector Field

**Curl** measures local tendency of a vector field and its flow lines to circulate around some axis.

The curl of a vector field is itself a vector field.

Setting; **F** : $\mathbb{R}^3 \to \mathbb{R}^3$ is our vector field

$\mathbf{F} = (F_1, F_2, F_3)$ so $\mathbf{F}(x, yz) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$

Formal Definition: curl $\mathbf{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

Mnemonic Device:

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Expand along first row:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$$

## Scalar Curl for Vector Fields in Plane

$\mathbf{F} = (F, G, 0)$ where$F(x, y)$ and $G(x, y)$are functions only of $x$ and $y$.

Then curl $\mathbf{F} = (0, 0, G_x - F_y)$

Note: Curl and Conservative Vector Field

Suppose $\mathbf{F} = (F, G, 0)$ is gradient field with $\mathbf{F} = \nabla f$.

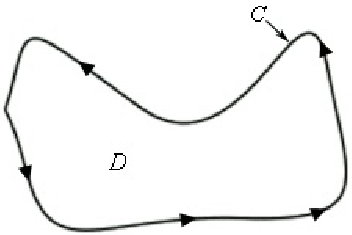
Then $F = f_x$ and $G = f_y$

In this case, Curl $\mathbf{F} = (0, 0, f_{yx} - f_{xy}) = (0, 0, 0)$

by Clairault's Theorem on Equality of Mixed Partials.

# Green's Theorem in the Plane

$$\iint_D \text{curl}\, \mathbf{F} = \int_\gamma \mathbf{F}$$




$D$ is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of $D$

$F$ and $G$ are continuously differentiable functions defined on $D$

Then

$$\int\int (G_x - F_y)dxdy = \int_\gamma (F, G)$$

where $\gamma$ is parametrized so it is traced once with $D$ on the left.

## Application of Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$

Example $\mathbf{F}(x,y) = (0,x)$ implies $curl| : \mathbf{F} = 1 - 0 = 1$

Hence $\iint_D \text{curl } \mathbf{F} = \iint_D 1 =$ area of $D$

Green's Theorem enables us to find the area of a planar region if we can develop a parametrization of its boundary.

Example Consider the unit disk $D$ of radius $r$ centered at origin.

Let $g(t) = (r\cos t, r\sin t), 0 \leq t \leq 2\pi$

So $g'(t) = (r\sin t, r\cos t)$

and $\mathbf{F}(g(t)) = (0, r\cos t)$

Then $\mathbf{F}(g(t)) \cdot g'(t) = r^2\cos^2 t\, dt$

Thus area of $D = \iint_D 1 = \iint_D \text{curl}\mathbf{F} = \int_\gamma \mathbf{F} = \int_0^{2\pi} r^2\cos^2 t\, dt$

$\int_0^{2\pi} r^2\cos^2 t\, dt = r^2\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = \frac{r^2}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2=\pi} = \pi r^2$

## Using Green's Theorem

(1) Compute $\iint_D \text{curl}\, \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \text{curl}\, \mathbf{F}$

## Using Green's Theorem

Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \text{curl}\,\mathbf{F}$

Example Let $\mathbf{F}(x,y) = (\frac{1}{y}\cos\frac{x}{y}, -\frac{x}{y^2}\cos\frac{x}{y})$

Compute $\int_\gamma \mathbf{F}$ as $\iint_D (G_x - F_y)$

Here $G_x = (-\frac{x}{y^2})_x \cos\frac{x}{y} + -\frac{x}{y^2}(\cos\frac{x}{y})_x$

$= -\frac{1}{y^2}\cos\frac{x}{y} - \frac{x}{y^2}(-\sin\frac{x}{y})(\frac{1}{y})$

$= -\frac{1}{y^2}\cos\frac{x}{y} + \frac{x}{y^3}(\sin\frac{x}{y})$

Similarly, $F_y = -\frac{1}{y^2}\cos\frac{x}{y} + \frac{1}{y}(-\sin\frac{x}{y})(\frac{-x}{y^2})$

$= -\frac{1}{y^2}\cos\frac{x}{y} + \frac{x}{y^3}(+\sin\frac{x}{y})$

So $G_x - F_y = 0$.

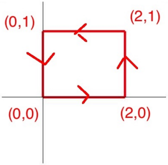
Hence $\int_\gamma \mathbf{F} = 0$

## Example

Find

$$\int_\gamma (1+10xy+y^2)\,dx+(6xy+5x^2)\,dy = \int_\gamma (1+10xy+y^2, 6xy+5x^2)$$

where $\gamma$ is boundary of the rectangle with vertices (0,0), (2,0), (2,1), and (0,1).




Note: Direct Computation requires 4 integrals.

$$F(x,y) = 1+10xy+y^2. \quad G(x,y) = 6xy+5x^2$$

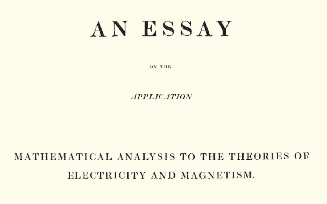
$$F_y = 10x+2y \qquad . \; G_x = 6y+10x$$

$$G_x - F_y = 6y+10x-10x-2y = 4t$$

$$\int_\gamma \mathbf{F} = \iint_D \text{curl}\,\mathbf{F} = \int_0^2 \int_0^1 4y\,dy\,dx = \int_0^2 \left[2y^2\right]_0^1 = \int_0^2 2dx = 4$$

George Green
1793 – 1841

Mikhail Ostrogradsky
1801 – 1861



AN ESSAY

ON THE

*APPLICATION*

MATHEMATICAL ANALYSIS TO THE THEORIES OF

ELECTRICITY AND MAGNETISM.

# Gauss' Theorem

$$\text{Green: } \iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$

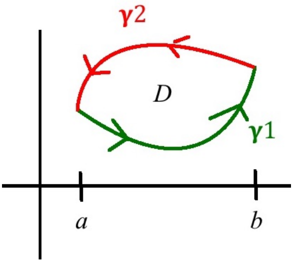
If $\mathbf{F} = (F_1, F_2)$ then curl $\mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

Apply Green's Theorem to $\mathbf{H} = (-G, F)$ where $\mathbf{F} = (F, G)$
$\int_\gamma \mathbf{H} = \iint_D$ curl $(F_x - (-G_y)) = \iint_D (F_x + G_y) = \iint_D$ div $\mathbf{F}$ On the other hand, $\int_\gamma \mathbf{H} = \int_a^b \mathbf{H} \cdot \mathbf{g'} = \int_a^b (-G, F) \cdot (g_1', g_2')$
$\int_a^b (-G, F) \cdot (g_1', g_2') = \int_a^b -Gg_1' + Fg_2' = \int_a^b (F, G) \cdot (g_2', -g_1')$
Observe $(g_2', -g_1') \cdot (g_1', g_2') = g_1' g_2' - g_1' g_2' = 0$
So $(g_2', -g_1')$ is orthogonal to the tangent vector so it is a normal vector $\mathbf{N}$.
Thus $\int_\gamma \mathbf{H} = \int_a^b (F, G) \cdot (g_2', -g_1') = \int_a^b (F, G) \cdot \mathbf{N} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$

Putting everything together: $\boxed{\iint_D \text{div } \mathbf{F} = \int_\gamma \mathbf{H} = \int_\gamma \mathbf{F} \cdot \mathbf{N}}$

## Proof of Green's Theorem in an Elementary Case

Case : Boundary of $D$ is made up of the graphs of two functions defined on interval $[a, b]$.




Ingredients:

Vector Field $\mathbf{F} = (F, G) = (F, 0) + (0, G)$

$\gamma_1 =$ image of $g_1$

$\gamma_2 =$ image of $g_2$

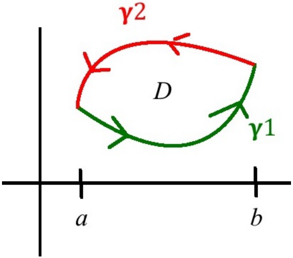
Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F,0) + (0,G)]$

Will show $\iint_D -Fy = \int_\gamma (F,0)$

We tackle the line integral first. Start with $\gamma_1$



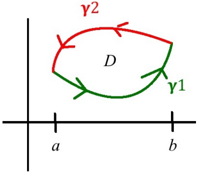

We can parametrize $\gamma_1$ by a function $g(t) = (t, \phi(t))$ for $a \leq t \leq b$

Then $g'(t) = (1, \phi_1'(t))$

Now $(F,0) \cdot g'(t) = (F,0) \cdot (1, \phi_1'(t)) = F = F(t, \phi_1(t))$

so $\int_{\gamma_1} (F,0) = \int_a^b F(t, \phi_1(t))\, dt$

Now we take up $\gamma_2$




Consider Parametrization of $\gamma_2$ as $g(t) = (t, \phi_2(t)), a \leq t \leq b$. This would actually traces out $\gamma_2$ in the opposite direction. It is the parametrization of $-\gamma_2$

Again we have $g'(t) = (1, \phi_2^{'})$ and $(F, 0) \cdot g'(t) = F(t, \phi_2(t))$

so $\int_{-\gamma_2}(F, 0) = \int_a^b F(t, \phi_2(t))$.

Thus $\int_{-\gamma_2}(F, 0) = -\int_{\gamma_2} = -\int_a^b F(t, \phi_2(t))$.

Finally, $\int_{\gamma}(F, 0) = \int_{\gamma_1}(F, 0) + \int_{\gamma_2}(F, 0)$

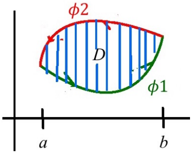
$= \int_a^b F(t, \phi_1(t))\, dt - \int_a^b F(t, \phi_2(t))\, dt$

$$\boxed{\int_{\gamma}(F, 0) = \int_a^b F(t, \phi_1(t)) - F(t, \phi_2(t))\, dt}$$

Goal: Show $\iint_D -Fy = \int_\gamma (F,0)$

So far: $\int_\gamma (F,0) = \int_a^b F(t,\phi_1(t)) - F(t,\phi_2(t))\,dt$

**Now turn to the curl part:**




$$\iint_D -Fy = -\iint_D F_y = \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} -Fy(x,y)\,dy\,dx$$

$$= -\int_a^b \left[F(x,\phi_2(x)) - F(x,\phi_1(x)\right]\,dx$$

$$= -\int_a^b \left[F(t,\phi_2(t)) - F(t,\phi_1(t)\right]\,dt(\text{ let } t=x)$$

$$= \int_a^b \left[F(t,\phi_1(t)) - F(t,\phi_2(t)\right]\,dt$$

## Conservative Vector Fields

**F** is continuously differentiable vector field in the plane
**F** : $\mathbb{R}^2 \to \mathbb{R}^2$ with **F**$(x, y) = (F(x, y), G(x, y))$ where $F$ and $G$ are each real-valued functions.

Here curl **F** is a real-valued function $G_x - F_y$

Green's Theorem: $\int_D$ curl **F** $= \int_\gamma$ **F**

## Three Important Properties of Vector Fields

**A**: **F** is **CONSERVATIVE** means **F** $= \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

**B**: **F** is **IRROTATIONAL** means curl **F** $= 0$

**C**: **F** is **PATH INDEPENDENT** means $\int_{\gamma_1}$ **F** $= \int_{\gamma_2}$ **F** for any paths $\gamma_1$ and $\gamma_2$ from **a** to **b** where **a** and **b** are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

**A** implies **B**

**A** **F** is **CONSERVATIVE**means **F** $= \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

**B** **F** is **IRROTATIONAL** means curl **F** $= 0$

Suppose **F** is Conservative

Then $(F, G) = \mathbf{F} = \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Then $G_x = f_{yx}$ and $F_y = f_{xy}$

so curl $\mathbf{F} = G_x - F_y = f_{yx} - f_{xy} = 0$

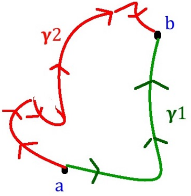
by equality of mixed partials.

**B** implies **C** will follow from Green's Theorem

**B** **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$

**C** **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths $\gamma_1$ and $\gamma_2$ from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and $\gamma_1$ and $\gamma_2$ two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**




and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let $D$ be= the enclosed region.

By Green's Theorem $\int_{\gamma} \mathbf{F} = \iint_D$ curl $\mathbf{F} = \iint_D 0 = 0$

Thus $0 = \int_{\gamma} \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$

Hence $\int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$

**C** implies **A**

**C** **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths $\gamma_1$ and $\gamma_2$ from **a** to **b** where **a** and **b** are any points in the plane.

**A** **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$