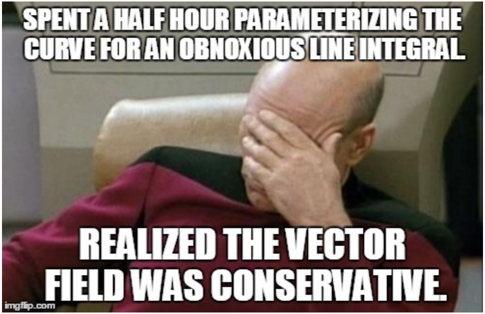
MATH 223: Multivariable Calculus



Class 33: Monday, May 5, 2025

Notes on Assignment 29
Assignment 30
Surface Integrals

Rough Weights for Course Components

Exam1	20%
Exam 2	20%
Exam 3	20%
Final Exam	30%
Project	10 %

Final Exam:
MATH 223A: Thursday, May 15 7 PM –10 PM

Announcements

Location Problem Due Friday, May 9

Today

More About Conservative Vector Fields

Surface Integrals

Conservative Vector Fields

F is continuously differentiable vector field in the plane
F : $\mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{F}(x, y) = (F(x, y), G(x, y))$ where F and G are each real-valued functions.

Here curl **F** is a real-valued function $G_x - F_y$

Green's Theorem: $\int_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$

Three Important Properties of Vector Fields

- **A** **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$
- **B** **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$
- **C** **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

A implies **B**

A **F** is **CONSERVATIVE** means **F** $= \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B **F** is **IRROTATIONAL** means curl **F** $= 0$

Suppose **F** is Conservative

Then $(F, G) =$ **F** $= \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Thus $G_x = f_{yx}$ and $F_y = f_{xy}$

so curl **F** $= G_x - F_y = f_{yx} - f_{xy} = 0$

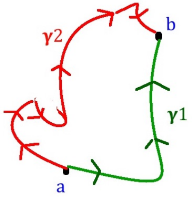
by equality of mixed partials.

B implies **C** will follow from Green's Theorem

B **F** is **IRROTATIONAL** means curl **F** $= 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and γ_1 and γ_2 two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**



and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let D be the enclosed region.

By Green's Theorem $\int_\gamma \mathbf{F} = \iint_D$ curl $\mathbf{F} = \iint_D 0 = 0$

Thus $0 = \int_\gamma \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$

Hence $\int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$

C implies **A**

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

A **F** is **CONSERVATIVE**means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

Idea:
Fix $\mathbf{x}_0$ in $\mathbb{R}^n$ and let $\mathbf{x}$ be arbitrary point in $\mathbb{R}^n$.
Let γ be a curve from $\mathbf{x}_0$ to $\mathbf{x}$.
Then $\int_\gamma \mathbf{F}$ will be a function of $\mathbf{x}$ whose gradient is $\mathbf{F}$.

Theorem Let $\mathbf{F}$ be a continuous vector field defined in a polygonally connected open set D of $\mathbb{R}^n$. If the line integral $\int_\gamma \mathbf{F}$ is independent of piecewise smooth path γ from $\mathbf{x}_0$ to $\mathbf{x}$ in D, then if $f(\mathbf{x}) = \int_\gamma \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.

Example $\mathbf{F}(x, y) = (3x^2 + y, e^y + x)$

Here $\mathbf{F} = (F, G)$ so $F(x, y) = 3x^2 + y, G(x, y) = e^y + x$

Hence $F_y = 1, G_x = 1$ so curl $\mathbf{F} = G_x - F_y = 0$

Let's build f so its gradient $\nabla f = (f_x, f_y) = (3x^2 + y, e^y + x)$

We need $f_x = 3x^2 + y$ so do "partial integration with respect to x":

$f(x) = x^3 + yx + g(y)$. [Why is there $g(y)$?]

Then $f_y = 0 + x + g'(y)$ which should equal $x + e^y$ so need

$$g'(y) = e^y$$

which we can get by letting $g(y) = e^y$.

Hence we can choose $f(x, y) = x^3 + yx + e^y + C$.

Let's build the potential function in a different way using the theorem with $\mathbf{F}(x, y) = (3x^2 + y, e^y + x)$

Pick $\mathbf{x}_0 = (0, 0)$ and let $\mathbf{x} = (x, y)$ be an arbitrary point. Choose the straight line between them as the path γ with parametrization $g(t) = (xt, yt), 0 \leq t \leq 1$ so $g'(t) = (x, y)$

Then $\mathbf{F}(g(t)) = F(xt, yt) = (3x^2t^2 + yt, e^{yt} + xt)$

so $\mathbf{F}(g(t)) \cdot g'(t) = (3x^2t^2 + yt, e^{yt} + xt) \cdot (x, y)$

$= 3x^3t^2 + xyt + ye^{yt} + xyt = 3x^3t^2 + 2xyt + ye^{yt}$

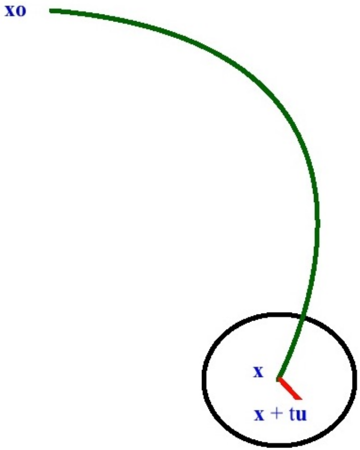
Now $\int_\gamma \mathbf{F} = \int_0^1 (3x^3t^2 + 2xyt + ye^{yt})\, dt$

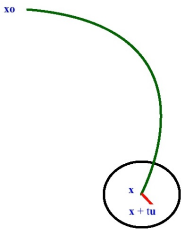
$= \left[x^3t^3 + xyt^2 + e^{yt}\right]_{t=0}^{t=1}$

$= (x^3 + xy + e^y) - (0 + 0 + 1) = x^3 + xy + e^y - 1$

Theorem Let **F** be a continuous vector field defined in a polygonally connected open set D of $\mathbb{R}^n$. If the line integral $\int_\gamma$ **F** is independent of piecewise smooth path γ from $\mathbf{x}_0$ to $\mathbf{x}$ in D, then if $f(\mathbf{x}) = \int_\gamma \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.

f





Let g be parametrization of line segment from $\mathbf{x}$ to $\mathbf{x} + t\mathbf{u}$ so $g(v) = \mathbf{x} + v\mathbf{u}, 0 \leq v \leq t$ and $g'(v) = \mathbf{u}$

$$f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}+t\mathbf{u}} \mathbf{F} - \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} = \int_{\mathbf{x}}^{\mathbf{x}+t\mathbf{u}} \mathbf{F}(\mathbf{x} + v\mathbf{u})$$
$$= \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u}\, dv$$

To find $\frac{\partial f}{\partial x_j}(\mathbf{x})$, let $\mathbf{u}$ be unit vector $\mathbf{e}_j = (0, 0, .. . , 1, 0, 0. . . 0)$ in the jth direction.

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u}\, dv$$

$$= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{e}_j) \cdot \mathbf{e}_j\, dv$$

But this last expression is the derivative of the integral with respect to t evaluated at $t = 0$ which is $\mathbf{F} \cdot \mathbf{e}_j = F_j(\mathbf{x})$ (Using Fundamental Theorem of Calculus)

Symmetry of Jacobian Matrix for Conservative Vector Field

Let **F** $= (F(x,y), G(x,y))$ be a conservative vector field in the plane which we can recognized by $G_x = F_y$

F' $= \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$ Note symmetry of Jacobian Matrix.

How do things generalize to higher dimensions?

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) = (yz^2 + \sin y + 3x^2, xz^2 + x\cos y + e^z, 2xyz + ye^z + \frac{1}{z})$$

$$\mathbf{F'} = \begin{pmatrix} 6x & z^2+\cos y & 2yz \\ z^2+\cos y & -x\sin y & 2xz+e^z \\ 2yz & 2xz+e^z & 2xy+ye^z-\frac{1}{z^2} \end{pmatrix}$$

To find f so that $\nabla f = \mathbf{F}$:

Step 1: integrate first component of **F** with respect to x:

$$f(x,y,z) = yz^2x + x\sin y + x^3 + G(y,z)$$

Step 2: Take derivative of trial f respect to y and set equal to second component of **F** :

$f_y = z^2x + x\cos y + 0 + G_y(y,z)$ must $= xz^2 + x\cos y + e^z$

Need $G_y(y,z) = e^z$ so choose $G(y,z) = e^z y + H(z)$

So far, $f(x,y,z) = yz^2x + x\sin y + x^3 + e^z y + H(z)$

Step 3:Take derivative of trial f respect to z and set equal to third component of **F** ;

$f_z(x,y,z) = 2xyz + 0 + 0 + e^z y + H'(z)$ must $= 2xyz + e^z y + \frac{1}{z}$

Need $H'(z) = \frac{1}{z}$ so choose $H(x) = \ln|z| + C$ Thus

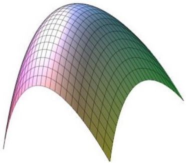
$$f(x,y,z) = f(x,y,z) = yz^2x + x\sin y + x^3 + e^z y + \ln|z| + C$$

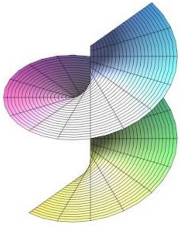
Theorem If **F** is a conservative vector field on $\mathbb{R}^n$ and is continuously differentiable, then the Jacobian matrix is symmetric.

Proof: Equality of mixed partials.

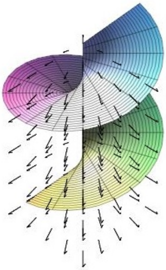
Theorem Suppose **F** is a continuously differentiable vector field on $\mathbb{R}^n$ whose Jacobian matrix is symmetric. Then **F** is conservative

Integrating Vector Fields Over Surfaces





$g(u,v) = [u, v, -2u^2 - 3v^2]$ $\quad$ $g(u,v) = [u\cos v, u\sin v, v]$



Smooth Curve γ	Smooth Surface S
$g : I$ in $\mathbb{R}^1 \to \mathbb{R}^n$	$g : D$ in $\mathbb{R}^2 \to \mathbb{R}^3$
Length $= \int_I \|g'(t)\|\, dt$	Area $\sigma(S) = \iint_D \|g_u \times g_v\| du dv$
Mass $= \int_I \mu(g(t))\|g'(t)\|\, dt$ Line Integral:	Mass $= \iint_D \mu \, d\sigma$ Surface Integral
$\int_\gamma \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t)\, dt$	$\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u,v)) \cdot (g_u \times g_v)$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$ is **flux** of $\mathbf{F}$ across S.

Surface Integral

Let g be a function from an interval $[t_0, t_1]$ into $\mathbb{R}^n$ with image γ and mu density at $g(t)$.

Then Mass of Wire $= \int_{t_0}^{t_1} \mu(t)|g'(t)|\, dt$

If $\mu \equiv 1$, then mass $=$ length of curve $\int_{t_0}^{t_1} |g'(t)|\, dt$

Generalize To Surfaces

Let D be region in plane and $g : D \to \mathbb{R}^3$ with $g(u, v) = (g_1, g_2, g_3)$ where each component function g_i is continuously differentiable.

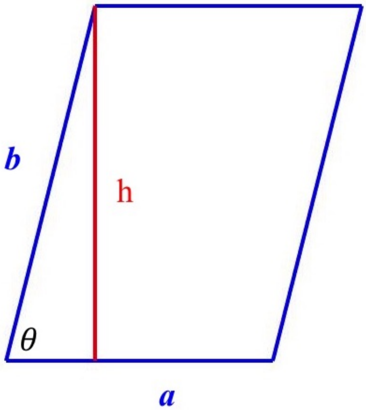
There are two natural tangent vectors: $g_u = \frac{\partial g}{\partial u}$ and $g_v = \frac{\partial g}{\partial v}$, These determine a tangent plane.

S is a **Smooth Surface** if these two vectors are linearly independent.

Note that $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$ is normal to the plane with

$|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}| = |\frac{\partial g}{\partial u}||\frac{\partial g}{\partial v}| \sin\theta$

$=$ Area of Parallelogram Spanned by the Vectors



$\sin\theta = \frac{h}{|\mathbf{b}|}$ so $h = |\mathbf{b}|\sin\theta$

Area of Parallelogram $=$ (Base)(Height) $= |\mathbf{a}||\mathbf{b}|\sin\theta$

$\mathbf{a} = g_u, \mathbf{b} = g_v$

$|g_u \times g_v| = |g_u||g_v|\sin\theta$

Surface Area

$$\sigma(S) = \iint_D |\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}|\, dudv = \iint_D |g_u \times g_v|\, dudv$$

If $\mu(g(u,v))$ is density, then mass $=$

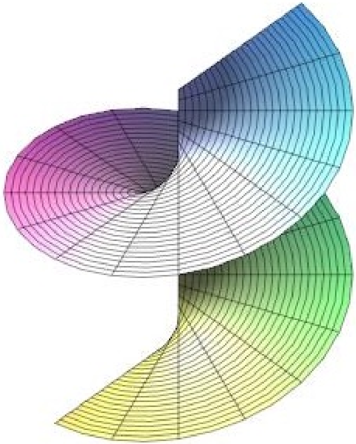
$$\iint_D \mu\, d\sigma = \iint_D \mu(g(u,v))|g_u \times g_v|\, dudv$$

Plotting Parametrized Surface in $Maple$:

$$plot3d([g1(u,v), g2(u,v), g3(u,v)], u = ..., v = ...)$$

Area of a Spiral Ramp

$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$



Area of a Spiral Ramp

$$g(u,v) = (u\cos v, u\sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u\sin v, u\cos v, 1)$$

$$g_u \times g_v = \text{det} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \left(\begin{vmatrix} \sin v & 0 \\ u\cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u\sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} \right)$$

$$= (\sin v, -\cos v, u)$$

Then $|g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$

Area $= \int_{v=0}^{v=3\pi} \int_{u=0}^{1} \sqrt{1+u^2}\, du\, dv$

If density is $\mu(\mathbf{x}) = u$, then

Mass $=$

$$\int_{v=0}^{v=3\pi} \int_{u=0}^{u=1} u(1+u^2)^{1/2}\, du\, dv = \int_{v=0}^{v=3\pi} \left[\tfrac{1}{3}(1+u^2)^{3/2}\right]_0^1\, dv$$

$$= \int_{v=0}^{v=3\pi} \tfrac{1}{3}[2^{3/2} - 1^{3/2}]\, dv = 3\pi\tfrac{1}{3}[2^{3/2}-1] = \pi[2^{3/2}-1]$$