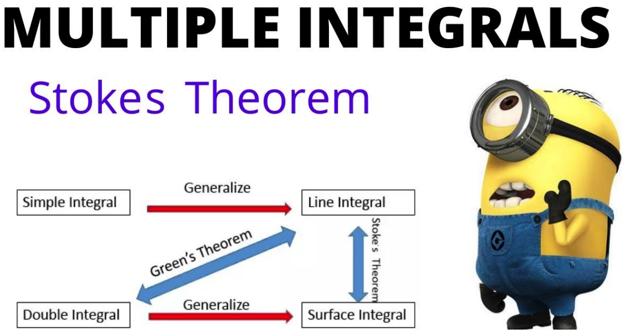
MATH 223: Multivariable Calculus



Class 35: Friday May 9, 2025

Handouts

Notes on Assignment 31

Assignment 32

Self Evaluation

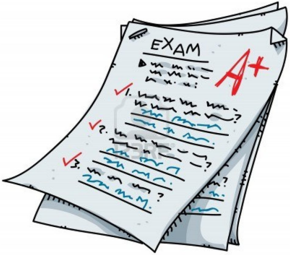
Peer Evaluation

MATH 223 Sticker

Announcements

- **Course Response Forms Monday**
  - Link: https://crfaccess.middlebury.edu/student/or **go/crf**
  - Available During Class Time Only
- **Final Examination**
  One Sheet of Notes

Final Exam



Next Thursday: 7 – 10 PM:
A - K: Warner 100
L - Z : Warner 010

Today:

Stokes' Theorem

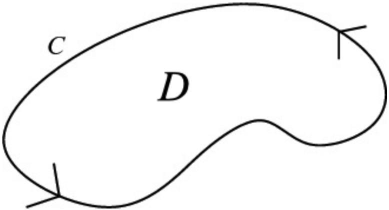
$$\int_S \text{curl } \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in $\mathbb{R}^3$

Vector Field Theorems

Plane

$\mathbf{F} : \mathcal{R}^2 \to \mathcal{R}^2$



D is 2-dimensional

$C = \partial D$ is 1-dimensional

Green – Ostrogradski	Gauss (Divergence)
$\int_D \text{curl } \mathbf{F} = \int_C \mathbf{F}$	$\int_D \text{div } \mathbf{F} = \int_C \mathbf{F} \cdot \mathbf{N}$

Positive Orientation

Setting: Let D be a plane region bounded by a curve traced out in a counterclockwise direction by some parametrization $h : \mathcal{R}^1 \to \mathcal{R}^2$ for $a \leq t \leq b$.

Let $S = g(D)$ be the image of D where $g : \mathcal{R}^2 \to \mathcal{R}^3$ so that S is a 2-dimensional surface in 3-space whose border γ corresponds to the boundary of D.

We say that γ inherits the **positive orientation** with respect to S. The composition $g(h(t))$ describes the border of S. Denote by ∂S the **positively oriented border** of S.

Vector Field in $\mathcal{R}^3$: $\mathbf{F}(\mathbf{x}) = (F(\mathbf{x}), G(\mathbf{x}), H(\mathbf{x}))$ where each of F, G, H is a real-valued function of 3 variables.

Curl $\mathbf{F}(\mathbf{x}) = (H_y(\mathbf{x}) - G_z(\mathbf{x}), F_z(\mathbf{x}) - H_x(\mathbf{x}), G_x(\mathbf{x}) - F_y(\mathbf{x}))$

Stokes' Theorem: Let S be a piece of smooth surface in $\mathcal{R}^3$, parametrized by a twice continuously differentiable function g. Assume that D, the parameter domain of g, is a finite union of simple regions bounded by a piecewise smooth curve. If $\mathbf{F}$ is a continuously differentiable vector field defined on S, then

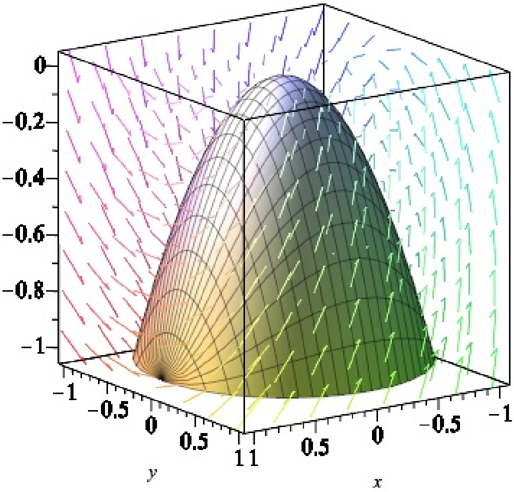
$$\int_S \text{Curl } \mathbf{F} \cdot dS = \int_{\partial S} F \cdot d\mathbf{x}$$

where ∂S is the positively oriented border of S.

[Note: If $\mathbf{F} = (F, G, 0)$ where F and F are independent of z, then Stokes' Theorem reduces to Green's Theorem. Thus Stokes generalizes Green.]

Example: Verify Stokes Theorem where

$$\mathbf{F}(x,y,z) = (z,x,y)$$
$$S : g(u,v) = (u,v,1-u^2-v^2), u^2+v^2 \leq 1.$$



Example: Verify Stokes Theorem where

$$\mathbf{F}(x, y, z) = (z, x, y)$$

$$S : g(u, v) = (u, v, 1 - u^2 - v^2), u^2 + v^2 \leq 1.$$

Parametrize ∂S by $(\cos t, \sin t, 0), 0 \leq t \leq 2\pi$.

Then $g(u, v) = (\cos t, \sin t, 0)$ and $g'(u, v) = (-\sin t, \cos t, 0)$

$$\mathbf{F}\left(g(u, v)\right) = (1 - u^2 - v^2, u, v) = (0, \cos t, \sin t)$$

$$\mathbf{F}\left(g(u, v)\right) \cdot g'(u, v) = (0, \cos t, \sin t) \cdot (-\sin t, \cos t, 0) = \cos^2 t$$

$$\int_{\partial S} \mathbf{F} = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi$$

Now $\int_S$ curl $\mathbf{F} = \int_S$ curl (z, x, y)

$$\text{curl } \mathbf{F} = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = (1-0, -(0-1), 1-0) = (1,1,1)$$

Thus we want to integrate (1,1,1) over S.
Here $g(u, v) = (u, v, 1 - u^2 - v^2)$
so $g_u = (1, 0, -2u), g_v = (0, 1, -2v)$
and $g_u \times g_v = (2u, 2v, 1)$ [work it out]

$$\int_S \text{curl } \mathbf{F} = \iint_D (1,1,1) \cdot (2u, 2v, 1)\, du\, dv = \iint_D 2u + 2v + 1\, du\, dv$$

which equals (using polar coordinates)

$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (2r\cos\theta + 2r\sin\theta + 1)\, r\; dr\; d\theta$$

$$\int_S \text{curl } \mathbf{F} = \iint_D (1,1,1)\cdot(2u,2v,1)\,du\,dv = \iint_D 2u+2v+1\,du\,dv$$

which equals (using polar coordinates)

$$\int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi}(2r\cos\theta + 2r\sin\theta + 1)\,r\,dr\,d\theta$$

$$= \int_{r=0}^{r=1}\left[2r^2\sin\theta - 2r^2\cos\theta + r\theta\right]_{\theta=0}^{\theta=2\pi}\,dr$$

$$= \int_{r=0}^{r=1} 2\pi r\,dr = 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=1} = \pi$$

Consequences of Stokes' Theorem

$$\int_S \text{curl } \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in $\mathbb{R}^3$

George Gabriel Stokes
August 13, 1819 – February 1, 1903
Stokes Biography

Interpretation of Curl

(1) The direction of curl $\mathbf{F}(\mathbf{x})$ is the axis about which $\mathbf{F}$ rotates most rapidly at $\mathbf{x}$. The length of curl $\mathbf{F}(\mathbf{x})$ is the maximum rate of rotation at $\mathbf{x}$.

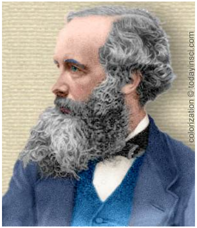
(2) **Maxwell's Equations**: curl $\mathbf{B} = \mathbf{I}$ where $\mathbf{I}$ is the vector current flow in an electrical conductor and $\mathbf{B}$ is the magnetic field which the current flow induces in the surrounding space.

Stokes' Theorem then yields **Ampere's Law**:

$$\int_S \mathbf{I} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{B} \cdot d\mathbf{x},$$

the total current flux across S is the circulation of the magnetic field around the border curve ∂S that encircles the conductor.

Definitions: A vector field $\mathbf{F}$ is **divergent-free** if div $\mathbf{F} = 0$ and $\mathbf{F}$ is **curl-free** if curl $\mathbf{F} = \mathbf{0}$.



James Clerk Maxwell (June 13, 1831 – November 5, 1879)

Maxwell Biography

Name	Equation	
	Integral form	Differential form
Faraday's law of induction	$\oint_c \vec{E}\cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$	$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère-Maxwell law	$\oint_c \vec{H}\cdot d\vec{l} = \iint_S \vec{J}\cdot d\vec{S} + \iint_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{S}$	$\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
Gauss' electric law	$\oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho\, dV$	$\nabla\cdot\vec{D} = \rho$
Gauss' magnetic law	$\oiint_S \vec{B}\cdot d\vec{S} = 0$	$\nabla\cdot\vec{B} = 0$

Theorem: A continuously differentiable gradient field has a symmetric Jacobian matrix.

Proof: If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$ for some real-valued function f. Then $\mathbf{F} = (f_x, f_y)$ so the Jacobian matrix is

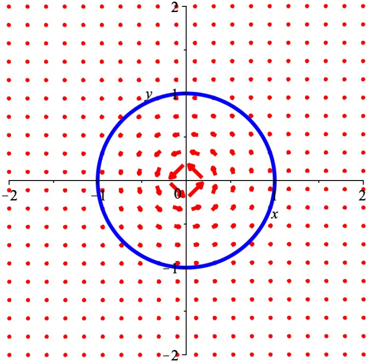
$$J = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

By Continuity of Mixed Partials, $f_{xy} = f_{yx}$ so J is symmetric. $\square$

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Has a Symmetric Jacobian But Is Not Conservative!

If $\mathbf{F}$ were conservative, then the line integral of $\mathbf{F}$ around any closed loop would be 0. Consider γ the unit circle as a loop running counterclockwise starting and ending at (1,0).



$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

γ: unit circle as a loop running counterclockwise starting and ending at (1.0).

We parametrize γ by $g(t) = (\cos t, \sin t), 0\pi$ so that $g'(t) = (-\sin t, \cos t)$ and

$$\mathbf{F}(g(t)) = \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) = (-\sin t, \cos t)$$

$$\mathbf{F}(g(t)) \cdot g'(t) = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = \sin^2 t + \cos^2 t = 1$$

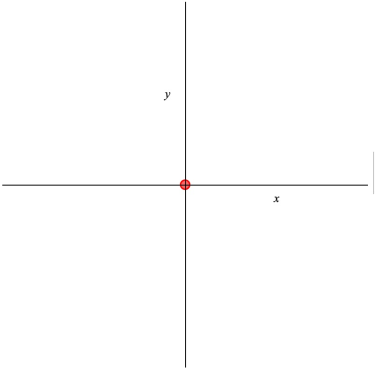
Thus $\int_\gamma \mathbf{F} = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$

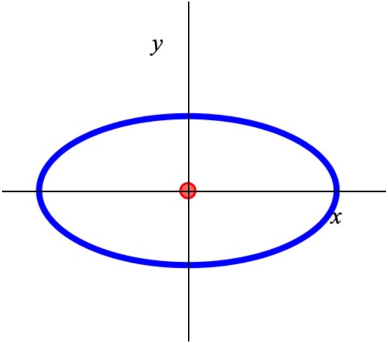
What is Wrong the Vector Field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)?$$

The Domain of the Vector Field

(Plane minus the Origin)
Is Not Simply Connected.



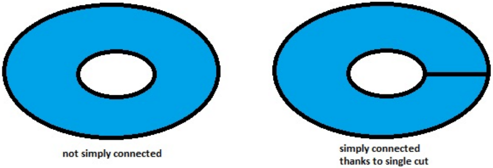


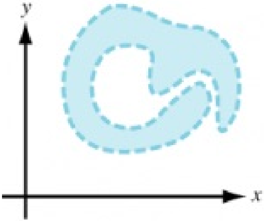
Simple Connectedness

A set B is **simply connected** if every closed curve in B can be continuously contracted to a point in such a way as to stay in B during the contraction. More precisely,

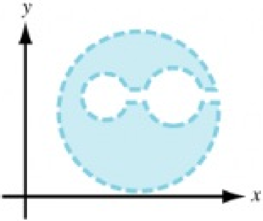
Definition: An open set B is **simply connected** if every piecewise smooth closed curve lying in B is the border of some piecewise smooth orientable surface S lying in B, and with parameter domain a disk in $\mathcal{R}^2$.

Theorem: Let $\mathbf{F}$ be a continuously differentiable vector field defined on an open set B in $\mathcal{R}^2$ or $\mathcal{R}^3$. If B is simply connected and curl F is identically zero in B, then $\mathbf{F}$ is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$

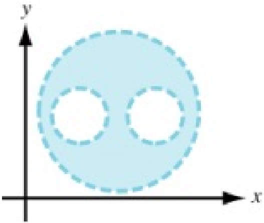


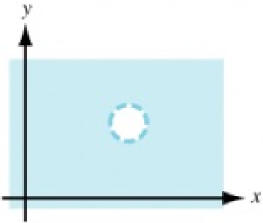


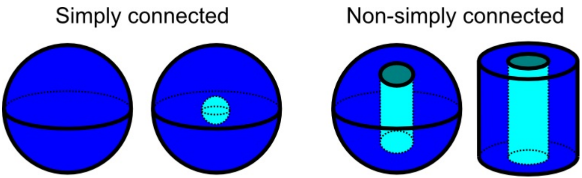
(a) A simply connected domain

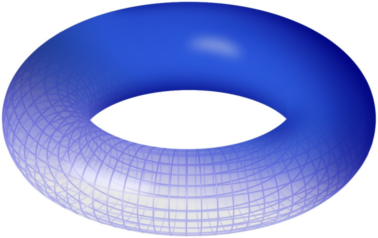


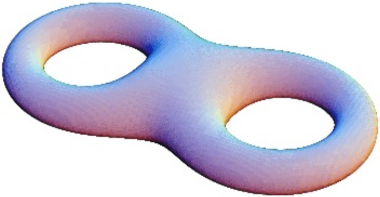
(b) A simply connected domain






Simply connected
Non-simply connected





Theorem: Let $\mathbf{F}$ be a continuously differentiable vector field defined on an open set B in $\mathcal{R}^2$ or $\mathcal{R}^3$. If B is simply connected and curl F is identically zero in B, then $\mathbf{F}$ is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$

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Proof: Let γ be a piecewise smooth closed loop in B. Because B is simply connected, there is a piecewise smooth surface S of which γ is the boundary.
By Stokes' Theorem

$$\int_\gamma \mathbf{F} = \int_S \text{curl } \mathbf{F} = \int_S \mathbf{0} = 0.$$

Thus $\mathbf{F}$ is path-independent and hence conservative. □

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Theorem: If the Jacobian matrix of a continuously differentiable vector field on a simply connected set is symmetric, then the vector field is conservative.

Proof: Suppose $\mathbf{F}$ is a vector field in $\mathcal{R}^3$ with

$\mathbf{F}(\mathbf{x}) = (F(\mathbf{x}), G(\mathbf{x}), H(\mathbf{x}))$ where $\mathbf{x} = (x, y, z)$

$$Jacobian = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix} \quad \text{with} \quad \begin{matrix} F_y = G_x \\ F_z = H_x \\ G_z = H_y \end{matrix}$$

curl $\mathbf{F} = (H_y - G_z, H_x - F_z, G_x - F_y) = (0, 0, 0) = \mathbf{0}$