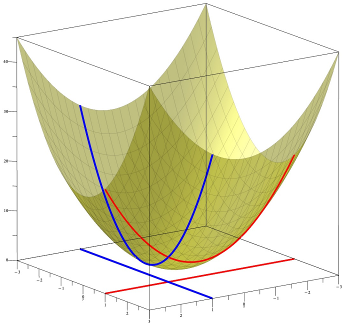
MATH 223: Multivariable Calculus



$$z = 2x^2 + 3y^2$$

Class 6: February 21, 2025

Notes on Assignment 5

Assignment 6

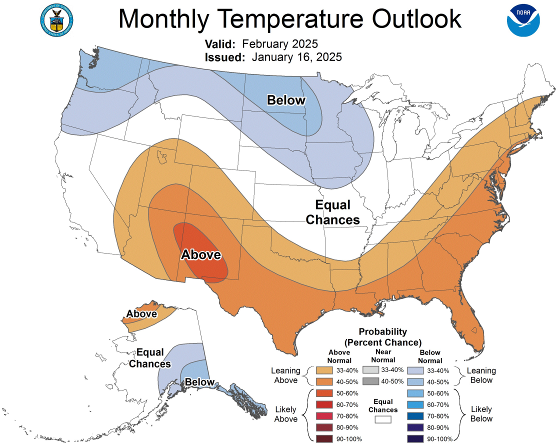
Some Surface Plotting Tools in MATLAB

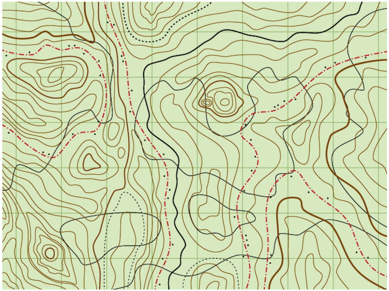
Level Curves

Given: output k (some constant)

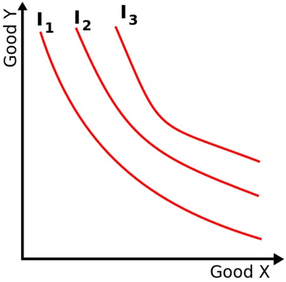
Find: All inputs Which Produce That Output

Examples Isotherms, Isobars, Isoclines, Indifference Curves





Topographic Map



Indifference Curve: Level Curve for Utility Function

Example:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Approach along line $y = mx$:

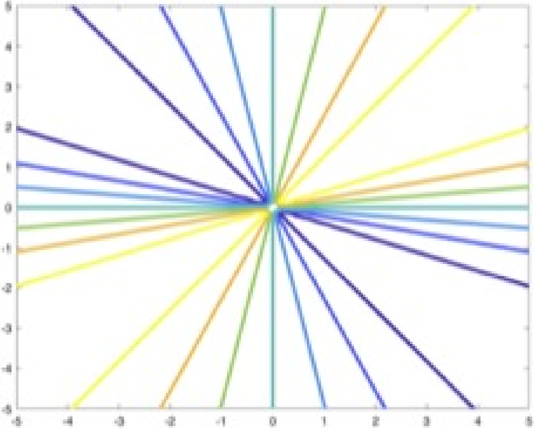
$$f(x,mx) = \frac{x(mx)}{x^2+2(mx)^2} = \frac{mx^2}{x^2+2m^2x^2} = \frac{mx^2}{x^2(1+2m^2)} = \frac{m}{1+2m^2}$$

m	$\frac{m}{1+2m^2}$
1	$\frac{1}{1+2} = \frac{1}{3}$
2	$\frac{2}{1+8} = \frac{2}{9}$
-1	$\frac{-1}{1+2} = -\frac{1}{3}$

Contours

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

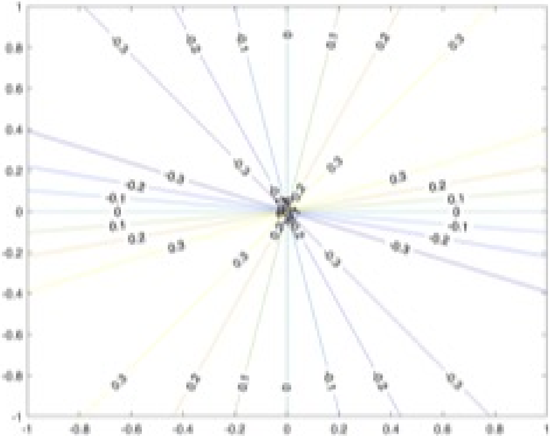
contour(x, y, f(x,y))



Contours

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

contour(X,Y,f(X,Y), 'ShowText','on')



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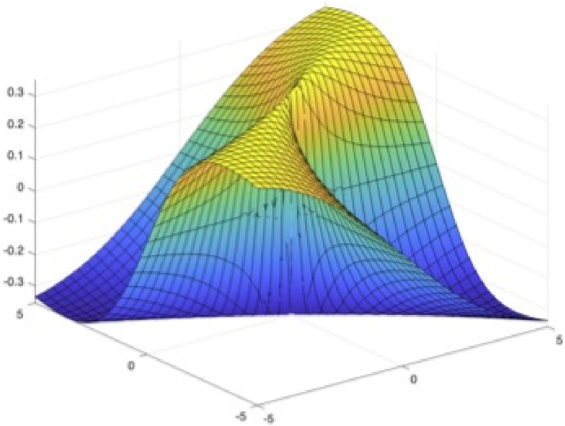

Derivative

$$f : \mathcal{R}^1 \to \mathcal{R}^1 : f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f : \mathcal{R}^1 \to \mathcal{R}^m : \mathbf{f}'(x) = \lim_{h \to 0} \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h}$$

What about $f : \mathcal{R}^n \to \mathcal{R}^1$?

$$f'(\mathbf{x}) \ \ \textbf{?=?} \ \ \lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})}{\mathbf{h}}$$

Major Problems

Division by vector **h** makes no sense.

Infinitely many ways $\mathbf{h} \to \mathbf{0}$.

Partial Solution
Consider 2 Special Ways for **h**

$\mathbf{h} = (t, 0)$	$\mathbf{h} = (0, t)$
$lim_{t\to 0} \frac{f(x+t,y)-f(x,y)}{t}$	$lim_{t\to 0} \frac{f(x,y+t)-f(x,y)}{t}$
$\partial f/\partial x, f_x, D[1](f)$ Partial Derivative With Respect to x Treat y as a constant Use usual rules of differentiation on x	$\partial f/\partial y, f_y, D[2](f)$ Partial Derivative With Respect to y Treat x as a constant Use usual rules of differentiation on y

Example: $f(x,y) = x^2y$ at point (3,4)

$$f_x(x,y) = 2xy \qquad f_y(x,y) = x^2$$
$$f_x(3,4) = 2 \times 3 \times 4 = 24 \qquad f_y(3,4) = 3^2 = 9$$

$$\begin{aligned}
f_x(x,y) &= \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t} \\
&= \lim_{t\to 0} \frac{(x+t)^2y - x^2y}{t} \\
&= \lim_{t\to 0} \frac{(x^2+2xt+t^2)y - x^2y}{t} \\
&= \lim_{t\to 0} \frac{x^2y + 2xyt + t^2y - x^2y}{t} \\
&= \lim_{t\to 0} \frac{2xyt + t^2y}{t} \\
&= \lim_{t\to 0} (2xy + ty) \\
&= 2xy
\end{aligned}$$

Example: $f(x,y) = x^2 y$

$$
\begin{aligned}
f_y(x,y) &= \lim_{t \to 0} \frac{f(x, y+t) - f(x,y)}{t} \\
&= \lim_{t \to 0} \frac{x^2(y+t) - x^2 y}{t} \\
&= \lim_{t \to 0} \frac{x^2 y + + x^2 t - x^2 y}{t} \\
&= \lim_{t \to 0} \frac{x^2 t}{t} \\
&= \lim_{t \to 0} x^2 \\
&= x^2
\end{aligned}
$$

Example: $f(x, y) = x^2 y$

$$f_x x, y = 2xy$$

$$f_y(x, y) = x^2$$

The **gradient** vector is

$$\nabla f = (f_x, f_y)$$

Let $\mathbf{w} = (x+2t, y+3t)$ so $\mathbf{h} = t(2,3)$

$$\begin{aligned}
f_{\mathbf{w}}(x,y) &= \lim_{t\to 0} \frac{f(x+2t, y+3t) - f(x,y)}{t} \\
&= \lim_{t\to 0} \frac{(x+t)^2(y+3t) - x^2y}{t} \\
&= \lim_{t\to 0} \frac{(x^2+2xt+t^2)(y+3t) - x^2y}{t} \\
&= \lim_{t\to 0} \frac{12t^3 + 12t^2x + 4t^2y + 3tx^2 + 4txy + x^2y - x^2y}{t} \\
&= \lim_{t\to 0} \frac{12t^3 + 12t^2x + 4t^2y + 3tx^2 + 4txy}{t} \\
&= \lim_{t\to 0} (12t^2 + 12tx + 4ty + 3x^2 + 4xy) \\
&= 4xy + 3x^2
\end{aligned}$$

With $\mathbf{w} = (x + 2t, y + 3t) = (x, y) + t(2, 3)$,

$$f_{\mathbf{w}}(x, y) = 4xy + 3x^2 = (2xy, x^2) \cdot (2, 3) = \nabla f \cdot (2, 3)$$

With $\mathbf{w} = (x + 2t, y + 3t) = (x, y) + t(2, 3)$,

$$f_{\mathbf{w}}(x, y) = 4xy + 3x^2 = (2xy, x^2) \cdot (2, 3) = \nabla f \cdot (2, 3)$$

COINCIDENCE?