Some Applications Related To The Material of This Course

Below are a dozen or so exercises that show some of the range of applications of *Multivariable Calculus*. You should be able to solve these problems and similar ones by the end of the course. If applications like these appeal to you, then you will enjoy applying the tools and techniques you will be learning in our class.

1. **Temperature Lapses.** The normal lapse rate for temperature above the surface of the earth assumes a steady drop of 3ºF per 1000 feet of increase of elevation. Under this assumption, with ground temperature 32ºF, and assuming negligible air resistance, estimate the temperature at time *t* at the height of a projectile fired straight up with an initial speed of 300 feet per second. What is the minimum temperature attained?

2. **Airplane Headings.** An airplane pilot wishes to maintain a true course in the direction 240º with a ground speed of 400 miles per hour when the wind is blowing directly north at 50 miles per hour. Find the required airspeed and compass heading.

3. **Big Bertha.** In World War I, Paris was bombarded by guns from the unprecedented distance of 75 miles away, shells taking 186 seconds to complete their trajectories. Estimate the angle of elevation at which the gun was fired and the maximum height of the trajectory, assuming negligible air resistance. (During a substantial part of the trajectory the altitude was high enough that air resistance was negligible there).

4. **Fox and Rabbit**. Suppose that a rabbit runs with constant speed $v > 0$ on a circular path of radius *a*, and that a fox, also running with constant speed *v*, pursues the rabbit by starting at the center of the circle, always maintaining a position on the radius from the center to the rabbit. How long does it take the fox to catch the rabbit? What is the path of motion of the fox?

5. **Heat Loss** A rectangular building is to be built to contain a fixed volume *V*. Heat loss through the roof and walls is proportional to area, while heat loss through the floor is negligible. Heat loss through the roof material is 3 times as rapid as heat loss through the wall material. What dimensions will minimize heat loss?

6. **Ideal Gas Law.** According to the ideal gas law, the pressure *P*, volume *V* and temperature *T* of a confined gas are related by the formula $PV = kT$ for a constant *k*. Express *P* as a function of *V* and *T*, and describe the level curves associated with this function. What is the physical significance of these level curves?

7. **Drug Use.** If a drug is taken orally, the time *T* at which the largest amount of drug is in the bloodstream can be calculated using the half-life *x* of the drug in the stomach and the half-life *y* of the drug in the bloodstream. For many common drugs (such as penicillin), *T* is given by $T = \frac{xy(\ln x - \ln y)}{(x - y)\ln 2}$. For a certain drug, $x = 30$ minutes and $y = 1$

hour. If the maximum error in estimating each half-life is ±10%, find the maximum error in the calculated value of *T*.

8. **A Toddler's Surface Area.** At age 2 years, a typical boy is 86 cm tall, weighs 13 kg and is growing at a ate of 9 cm/year and 2 kg/year. Use the Dubois and DuBois surface area formula $S = 0.007184x^{0.425}y^{0.725}$ for weight *x* and height *y* to estimate the rate at which the body surface area is growing.

9. **Hardy-Weinberg** Three alleles (alternative forms of a gene) A, B and O determine the four human blood types: A (AA or AO), B (BB or BO), O (OO) and AB. The *Hardy-Weinberg* law asserts that the proportion of individuals in a population who carry two different alleles is given by the formula $P = 2pq + 2pr + 2rq$ where *p*, *q*, and *r* are the proportions of alleles A, B, and O, respectively, in the population. Show that P must be les than or equal to 2/3

10. **Cobb-Douglas Production Functions** If *x* units of capital and *y* units of labor are required to manufacture *f(x,y)* units of a certain commodity, the Cobb-Douglas production function is defined by $f(x, y) = kx^a y^b$ where *k* is a constant and *a* and *b* are positive numbers *a* + *b* = *1*. Suppose that $f(x, y) = x^{1/5}y^{4/5}$ and that each unit of capital costs *C* dollars and each unit of labor costs *L* dollars. If the total amount available for these costs is *M* dollars, *so xC* + *yL* = *M*, how many units of capital and labor will maximize production?

11. **Volume of the earth.** Because of rotation, the earth is not perfectly spherical bt is slightly flattened at the poles, with a polar radius of 6356 km and an equatorial radius of 6378 km. As a result, the shape of the earth's surface can be approximated by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with *a* = *b* = 6378 and *c* = 6356. Estimate the volume of the earth.

12. **Gypsy Moths on the Loose.** We can model the spread of an aerosol in the atmosphere by a combination of wind and diffusion using a function of three variables. The concentration of the aerosol reaches a steady state independent of time at a point **x** given by $S(\mathbf{x}) = \dfrac{Q}{2\pi v_0 \sigma_y \sigma_z x^{2-n}} e^{-\left(y^2/\sigma_y^2 + z^2/\sigma_z^2\right)/2x^{2-n}}$ if the aerosol is released from the origin and the wind is blowing in the positive *x*-direction with velocity v_0. Here Q is the rate of release of the chemical and σ_y, σ_z and n are empirically determined constants. We measure distance in centimeters, velocity in centimeters per second, and concentration in parts per cubic centimeter. The values $\sigma_y = 0.4$, $\sigma_z = 0.2$ and $n = 0.25$ give good results for wind velocities less than 500 cm/s. Among other applications, we can model the diffusion of insect pheromones.

Pheromones are "odor" chemicals released by animals for chemical communication within a species. For a single female gypsy moth, Q is on the order of 3×10^{13} particles per cubic centimeter. Given that a male gypsy moth can detect as few as 10 0 particles per cubic cm, use the Lagrange multipliers method to determine the maximum distance downward from a female moth that a male moth can detect a female gypsy moth.

13. **Optimal Investment Policy.** An organization wants to determine an optimal allocation of investment between labor and capital stock over an extended period of time, beginning at time $t = 0$ and ending at time $t = T$ (which may be infinite). If s and e represent the fraction of the output allocated to investment in capital stock and human labor, respectively, then we want to choose s and e so that we maximize $\int_O^T (1 - s - e)\, wf\, e^{-\gamma t}\, dt$. Here w represents fractional employment, $f(r/w)$ is the output per employed worker, r is the capital-labor ratio, and γ is the consumption rate. Note that we require $r'(t) = swf - (n+\delta)r$ where n is the rate of growth of population and δ is the rate of depreciation of capital and we also need $w'(t) = (e/n)wf - (n+\mu)w$ where μ is the death rate. Use Stokes's Theorem to find the optimal allocation plan.