ON PROBLEM SOLVING

A major part of your time in Calculus and other courses is devoted to solving problems. It is worth your while to develop sound techniques. Here are a few suggestions.

Think. Before plunging into a problem, take a moment to think. Read the problem again. Think about it. What are its essential features? Have you seen a problem like it before? What techniques are needed?

Try to make a rough estimate of the answer. It will help you understand the problem and will serve as a check against unreasonable answers. A car will not go 1,000 miles in 3 hours; a weight dropped from 10,000 feet will not hit the earth at 5 mph; the volume of a tank is not -275 gal.

Examine the data. Be sure you understand what is given. Translate the data into mathematical language. Whenever possible, make a clear diagram and label it accurately. Place axes to simplify computations. If you get stuck, check that you are using all the data.

Avoid sloppiness.

(a) *Avoid sloppiness in language.* Mathematics is written in English sentences. A typical mathematical sentence is "$y = 4x + 1$." The equal sign is the verb in this sentence; it means "equals" or "is equal to." The equal sign is not to be used in place of "and", nor as a punctuation mark.

Quantities on opposite sides of an equal sign must be equal.

Use short simple sentences. Avoid pronouns such as "it" and "which". Give names and use them. Consider the following example.

"To find the minimum of it, differentiate it and set it equal to zero, then solve it which if you substitute it, it is the minimum."

Better: "To find the minimum of $f(x)$, set its derivative $f\ '(x)$ equal to zero. Let x_0 be the solution of the resulting equation. Then $f(x_0)$ is the minimum value of $f(x)$."

(b) *Avoid sloppiness in computation.* Do calculations in a sequence of neat, orderly steps. Include all steps except utterly trivial ones. This will help eliminate errors, or at least make errors easier to find. Check any numbers used; be sure that you have not dropped a minus sign or transposed digits.

(c) *Avoid sloppiness in units.* If you start out measuring in feet, all lengths must be in feet, all areas in square feet, and all volumes in cubic feet. Do not mix feet and acres, seconds and years.

(d) *Avoid sloppiness in the answer.* Be sure to answer the question that is asked. If the problem asks for the maximum value of $f(x)$, the answer is not the point where the maximum occurs. If the problem asks for a formula, the answer is not a number.

EXAMPLE Find the minimum of $f(x) = x^2 - 2x + 1$.

Solution 1:

$$2x - 2$$
$$x = 1$$
$$1^2 - 2 \cdot 1 + 1$$
$$0$$

Unbearable. This is just a collection of marks on the paper. There is absolutely no indication of what these marks mean or of what they have to do with the problem. When you write, it is your responsibility to inform the reader what you are doing. Assume he is intelligent, but not a mind reader.

Solution 2:

$$\frac{df}{dx} = 2x - 2 = 0 = 2x = 2 = x = 1$$

$$= f(x) = 1^2 - 2 \cdot 1 + 1 = 0.$$

Poor. The equal sign is badly mauled. This solution contains such enlightening statements as "0 = 2 = 1," and it does not explain what the writer is doing.

<u>Solution 3</u>:

$$\frac{df}{dx} = 2x - 2 = 0, \quad 2x = 2, \quad x = 1.$$

This is better than Solution 2, but contains two errors. Error 1: The first statement, "$\frac{df}{dx} = 2x - 2 = 0$," muddles two separate steps. First the derivative is computed, then the derivative is equated to zero. Error 2: The solution is incomplete because it does not give what the problem asks for, the minimum value of f. Instead, it gives the point x at which the minimum is assumed.

<u>Solution 4</u>: The derivative of f is

$$f' = 2x - 2.$$

At a minimum, $\mathbf{f'} = 0$. Hence

$$2x - 2 = 0, \quad x = 1.$$

The corresponding value of f is

$$f(1) = 1^2 - 2 \cdot 1 + 1 = 0.$$

If $x > 1$, then $f'(x) = 2(x-1) > 0$, so f is increasing. If $x < 1$, then $f'(x) = 2(x-1) < 0$, so f is decreasing. Hence f is minimal at x = 1, and the minimum values of f is 0.

This solution is absolutely correct, but long. For homework assignments the following is satisfactory (check with your instructor):

<u>Solution 5</u>:

$$f'(x) = 2x - 2.$$

At min, $f' = 0$, $2x - 2 = 0$, $x = 1$. For $x > 1$, $f'(x) = 2(x-1) > 0$, f□: for $x < 1$, $f'(x) = 2(x-1) < 0$, f□.

Hence $x = 1$ yields min,

$$f_{min} = f(1) = 1^2 - 2 \cdot 1 + 1 = 0.$$

The next solution was submitted by a student who took a moment to think.

<u>Solution 6</u>:

$$f(x) = x^2 - 2x + 1 = (x-1)^2 \geq 0.$$

But

$$f(1) = (1-1)^2 = 0.$$

Hence the minimum value of $f(x)$ is 0.

- from *A First Course in Calculus*
by Flanders, Korfhage and Price