Sections 1.1 and 1.2:
Vectors in Two and Three Dimensions

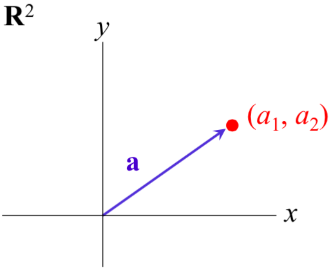
Algebraic and Geometric Points of View

Vectors

A vector in $\mathbb{R}^2$ is an ordered pair

$$\bar{a} = (a_1, a_2).$$

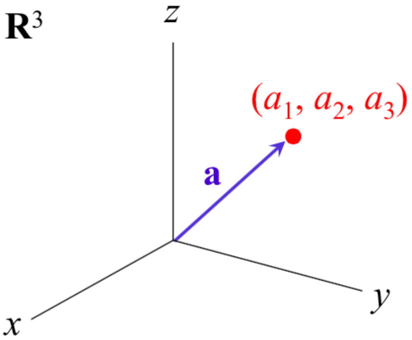
For example, $(2, 3)$, $(\pi, \sqrt{2})$.



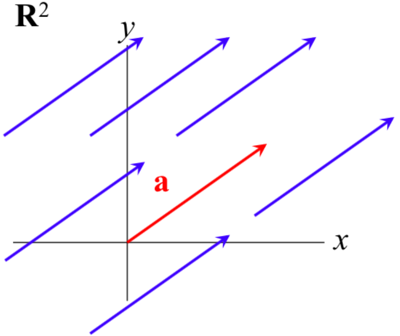
A vector in $\mathbb{R}^3$ is an ordered triple

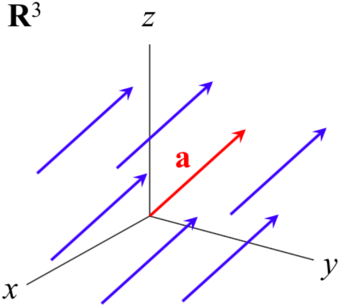
$$\bar{a} = (a_1, a_2, a_3).$$

For example, $(-1, 7, 32)$, $(3, -e^2, 0)$.



Unlike points, we think of vectors as mobile in space.



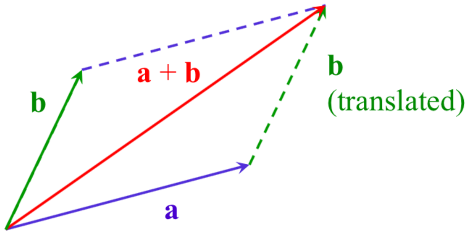


If $\overline{a} = (a_1, a_2, a_3)$, then the red vector represents the position vector of the *point* $P = (a_1, a_2, a_3)$.

There are two operations that we apply to vectors:

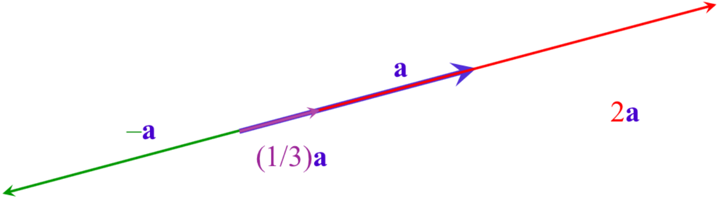
- Addition:

$$\bar{a} + \bar{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$



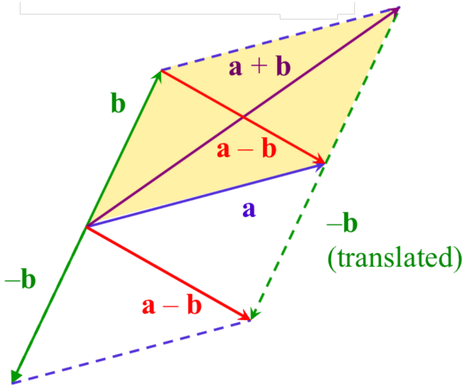
- and scalar multiplication:

$$k\bar{a} = k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$$



To subtract, consider

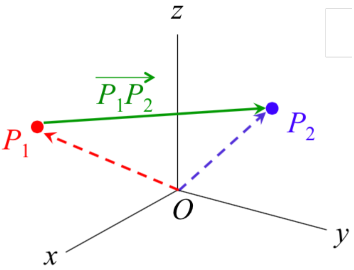
$$\bar{a} - \bar{b} = \bar{a} + (-\bar{b})$$



The displacement vector between two points is given by the difference of their position vectors.

$$\text{If } P_1 = (x_1, y_1, z_1) \text{ and } P_2 = (x_2, y_2, z_2) \text{ then}$$

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$



The Standard Basis Vectors

The standard basis vectors in $\mathbb{R}^2$ are:

$$\bar{i} = (1, 0) \qquad \bar{j} = (0, 1)$$

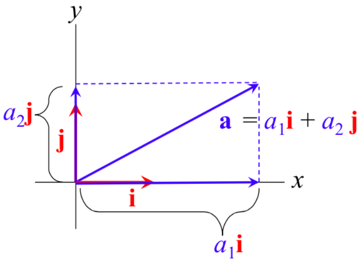
$$\text{so } \bar{a} = (a_1, a_2) = a_1\bar{i} + a_2\bar{j}.$$

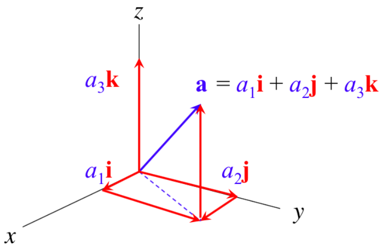
The standard basis vectors in $\mathbb{R}^3$ are:

$$\bar{i} = (1, 0, 0) \qquad \bar{j} = (0, 1, 0) \qquad \bar{k} = (0, 0, 1)$$

$$\text{so } \bar{a} = (a_1, a_2, a_3) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}.$$

Visually:





Parametric Equations of Lines

In $\mathbb{R}^2$, a line is determined by the equation

$$y = mx + b \qquad \text{a.k.a.} \;\; Ax + By = C.$$

In $\mathbb{R}^3$, however, the equation

$$Ax + By + Cz = D$$

determines a *plane*, not a line. (More on this soon.)

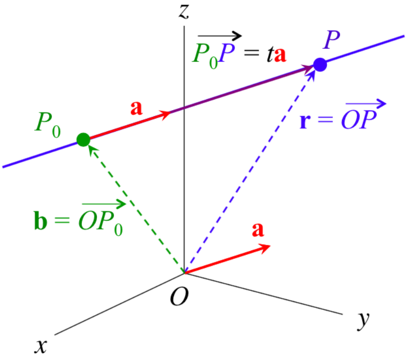
To describe a line (or any curve) in $\mathbb{R}^n$, use vector or parametric equations with one parameter, t, instead.

Given

- a specific point $P = (b_1, b_2, b_3)$ on the line and
- a vector $\overline{a} = (a_1, a_2, a_3)$ parallel to the line,

the position vector of a general point $\overline{r}$ on the line can be written

$$\overline{r}(t) = \overline{b} + t\overline{a}, \quad t \in \mathbb{R}$$



$$\overline{r}(t) = \overline{b} + t\overline{a}, \quad t \in \mathbb{R}.$$

- The expression above is called the vector equation of a line.

- The vectors $\overline{b}$ and $\overline{a}$ are *fixed*. The scalar t is *variable*, and is referred to as a parameter.

- In terms of components,

$$\overline{r}(t) = (x(t), y(t), z(t)) = (b_1 + ta_1, b_2 + ta_2, b_3 + ta_3)$$

which gives us the parametric equations of a line:

$$\begin{aligned} x(t) &= b_1 + ta_1 \\ y(t) &= b_2 + ta_2 \\ z(t) &= b_3 + ta_3. \end{aligned}$$

Example

Line between $(3, 1, 4)$ and $(2, 3, 7)$.

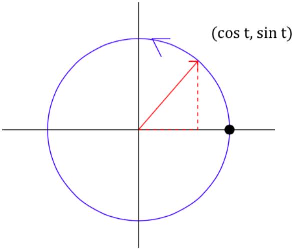
Parametric Equations of Curves

More generally, a set of parametric equations in $\mathbb{R}^n$ determines a curve in $\mathbb{R}^n$.

For example,

$$x(t) = \cos t$$
$$y(t) = \sin t, \qquad 0 \le t \le 2\pi$$

defines a circle in $\mathbb{R}^2$.



Important observation:

- The parametric equations for a line or curve are not unique.
- There is always more than one way to describe a particular curve using parametric equations.