Sections 1.3 and 1.4: The Dot and Cross Products

A Scalar Product and a Vector Product

The Dot and Cross Products

Both take in two vectors, $\bar{a}$ and $\bar{b}$.

- $\bar{a} \cdot \bar{b}$ is a scalar.
- $\bar{a} \times \bar{b}$ is a vector.

Each has geometric interpretations.

The Dot Product

If $\bar{a} = (a_1, a_2, a_3)$

and $\bar{b} = (b_1, b_2, b_3)$,

define $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

This definition works for vectors in $\mathbb{R}^n$ for any n:

If $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$,

then $\bar{a} \cdot \bar{b} = a_1 b_1 + \cdots + a_n b_n$.

Algebraic Properties of the Dot Product

Suppose that $\bar{a}$, $\bar{b}$, and $\bar{c}$ are vectors and $k \in \mathbb{R}$ is a scalar. Then

- $\bar{a} \cdot \bar{a} \geq 0$, and $\bar{a} \cdot \bar{a} = 0$ only when $\bar{a} = 0$. (Positive-definiteness)
- $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$. (Symmetry)
- $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$. (Linearity)
- $(k\bar{a}) \cdot \bar{b} = k(\bar{a} \cdot \bar{b}) = \bar{a} \cdot (k\bar{b})$. (Linearity)

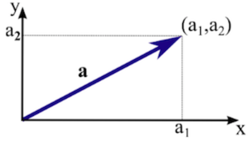
Note that symmetry implies that the dot product is linear in the first term as well.

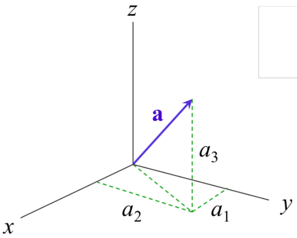
Geometric Properties of the Dot Product

The length, or norm, of a vector $\bar{a}$ is given by

$$\|\bar{a}\| = \sqrt{\bar{a}\cdot\bar{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

It's the Pythagorean Theorem:





A unit vector is a vector $\bar{a}$ such that $\|\bar{a}\| = 1$.

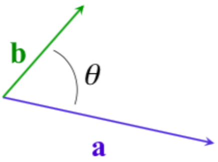
If $\bar{a} \neq \bar{0}$, then

$$\bar{u} = \frac{1}{\|\bar{a}\|}\bar{a}$$

is a unit vector.

Scaling a nonzero vector $\bar{a}$ to make it a unit vector is called normalizing $\bar{a}$.

The dot product can be used to detect angles between vectors.



Theorem

If $\bar{a}$ *and* $\bar{b}$ *are vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*, then*

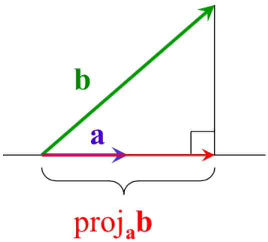
$$\bar{a} \cdot \bar{b} = \|\bar{a}\| \|\bar{b}\| \cos\theta$$

where $0 \leq \theta \leq \pi$*.*

Important note: Since $\cos\frac{\pi}{2} = 0$, $\bar{a} \perp \bar{b}$ if and only if $\bar{a} \cdot \bar{b} = 0$.

The Dot Product and Projections

We can use the dot product to calculate the projection of one vector onto another.



For any vectors $\bar{a}$ and $\bar{b}$ such that $\bar{a} \neq \bar{0}$,

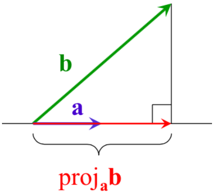
$$\text{proj}_{\bar{a}}\bar{b} = \left(\frac{\bar{a} \cdot \bar{b}}{\|\bar{a}\|}\right)\frac{\bar{a}}{\|\bar{a}\|} = \left(\frac{\bar{a} \cdot \bar{b}}{\bar{a} \cdot \bar{a}}\right)\bar{a}.$$

Why?

$$\text{proj}_{\bar{a}}\bar{b} = \left(\frac{\bar{a} \cdot \bar{b}}{\|\bar{a}\|}\right)\frac{\bar{a}}{\|\bar{a}\|}$$

- $\frac{\bar{a}}{\|\bar{a}\|}$ is a unit vector in the direction of $\bar{a}$.

- We need to show that $\frac{\bar{a} \cdot \bar{b}}{\|\bar{a}\|}$ gives the length of $\text{proj}_{\bar{a}}\bar{b}$.



Let θ be the angle between $\bar{a}$ and $\bar{b}$.

From trig class,

$$\|\bar{b}\| \cos\theta = \text{length of } \mathrm{proj}_{\bar{a}}\bar{b}.$$

But we also know that

$$\bar{a} \cdot \bar{b} = \|\bar{a}\|\|\bar{b}\| \cos\theta,$$

so

$$\text{length of } \mathrm{proj}_{\bar{a}}\bar{b} = \frac{\bar{a} \cdot \bar{b}}{\|\bar{a}\|}.$$

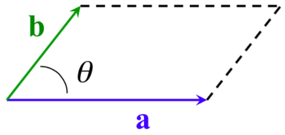
The Cross Product

This product is defined only for vectors in $\mathbb{R}^3$.

Given $\bar{a}$ and $\bar{b}$, to define $\bar{a} \times \bar{b}$, we need to define its

length and direction.

Length:



$$\|\bar{a} \times \bar{b}\| = \text{ area of parallelogram spanned by } \bar{a} \text{ and } \bar{b}.$$

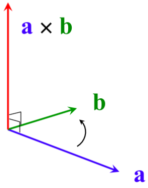
Note:

- If $\bar{a}$ or $\bar{b} = \bar{0}$, or if $\bar{a} \parallel \bar{b}$, then $\bar{a} \times \bar{b} = \bar{0}$.
- By trig, $\|\bar{a} \times \bar{b}\| = \|\bar{a}\| \|\bar{b}\| \sin \theta$.

Direction:

$$\bar{a} \times \bar{b} \text{ is orthogonal to both } \bar{a} \text{ and } \bar{b}\text{, and}$$

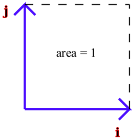
$$\{\bar{a}, \bar{b}, \bar{a} \times \bar{b}\} \text{ forms a right-handed system.}$$



These two pieces of information completely determine the direction of $\bar{a} \times \bar{b}$.

Example

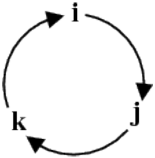
Cross product of standard basis vectors.



By properties of cross product

$$\bar{i} \times \bar{j} = \bar{k}.$$

In general



Algebraic Properties of the Cross Product

Suppose that $\bar{a}$, $\bar{b}$, $\bar{c}$ are vectors and k is a scalar. Then

- $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$.
- $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$.
- $(\bar{a} + \bar{b}) \times \bar{c} = \bar{a} \times \bar{c} + \bar{b} \times \bar{c}$.
- $(k\bar{a}) \times \bar{b} = k(\bar{a} \times \bar{b}) = \bar{a} \times (k\bar{b})$.

These properties can be proven using the geometric definition of the cross product.

Computing the cross product is a lot like taking a determinant.

If $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$, then

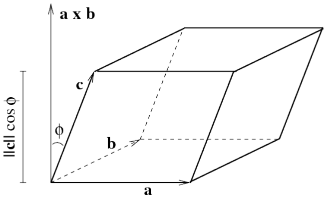
$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\bar{i} - (a_1 b_3 - a_3 b_1)\bar{j} + (a_1 b_2 - a_2 b_1)\bar{k}.$$

Example

$\bar{j} \times \bar{i}$

$$\bar{j} \times \bar{i} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = (0 - 0)\bar{i} - (0 - 0)\bar{j} + (0 - 1)\bar{k} = -\bar{k}.$$

Consider the parallelepiped:



Then

$$\begin{aligned} \text{volume} &= (\text{area base})(\text{height}) \\ &= \|\bar{a} \times \bar{b}\| \|\bar{c}\| |\cos\phi| \qquad \text{(by trig)} \\ &= |(\bar{a} \times \bar{b}) \cdot \bar{c}|. \end{aligned}$$

Since the volume of the parallelepiped is constant, the absolute value of the scalar triple product $(\bar{a} \times \bar{b}) \cdot \bar{c}$ is invariant under rearrangement.