Math 223 - Multivariable Calculus
Practice Exam 1

Name: solutions

Please be sure to neatly **show and explain all of your work** and clearly label your answers. Other than your index card, this is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

Signed:

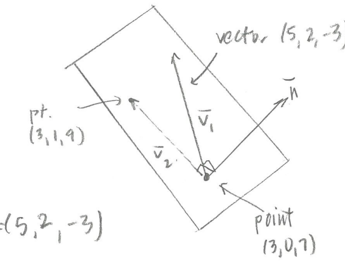
Problem	Points
1	/12
2	/10
3	/12
4	/12
5	/12
6	/10
7	/12
Total	/80

1. Determine the equation of the plane containing the line

$$\bar{r}(t) = (3+5t, 2t, 7-3t)$$

and the point $(3, 1, 9)$. Show all work.

$\bar{r}(t) = \underbrace{(3,0,7)}_{\text{pt.}} + t\underbrace{(5,2,-3)}_{\text{direction vector}}$



We have one vector in the plane: $\bar{v}_1 = (5,2,-3)$

To find another, take the displacement vector b/w $(3,1,9)$ and $(3,0,7)$:

$\bar{v}_2 = (3-3, 1-0, 9-7) = (0,1,2)$

The normal to the plane is given by $\bar{v}_1 \times \bar{v}_2$:

$$\begin{array}{r} (5,2,-3) \\ \times\ (0,1,2) \\ \hline \end{array}$$

$\bar{n} = (4-(-3), 0-10, 5-0) = (7,-10,5)$

Using $(3,1,9)$ as our point in the plane, the plane is given by the equation

* (don't forget this part)

$(7,-10,5) \cdot (x-3, y-1, z-9) = 0$

$\Rightarrow 7(x-3) - 10(y-1) + 5(z-9) = 0$

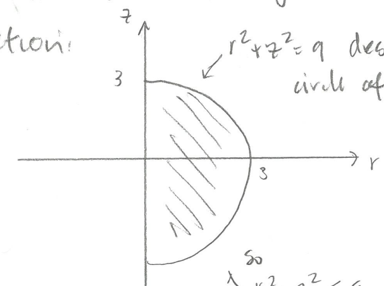
$\rightarrow \boxed{7x - 10y + 5z = 56}$

-21
-10
11
+45
56

2. Please sketch (and describe) the region in $\mathbb{R}^3$ given by the inequalities

$$r^2 + z^2 \le 9,$$
$$0 \le \theta \le \tfrac{\pi}{2}.$$

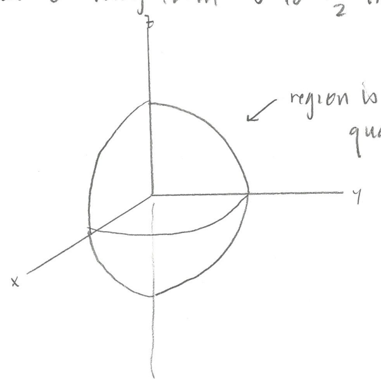
Since the first inequality involves only r and z, consider a θ-cross section:



$r^2+z^2=9$ describes a circle of radius 3

so $r^2+z^2 \le 9$ gives a half disk (we assume $r \ge 0$)

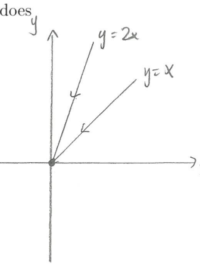
Now, we let θ vary from 0 (xz-plane) to $\frac{\pi}{2}$ (yz-plane) in $\mathbb{R}^3$, rotating the half disk.



region is a solid quarter sphere

3. (a) Please compute the following limit or explain clearly why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{y}{x}$$



This limit does not exist. [$\frac{y}{x}$ is not defined at (0,0) so we can't plug in]

Traveling to (0,0) along the line $y=x$, for all points on this line $\frac{y}{x} = \frac{x}{x} = 1$, so the limit along this line is 1.

Traveling to (0,0) along the line $y=2x$, however, for all points on this line $\frac{y}{x} = \frac{2x}{x} = 2$, so the limit along this line is 2.

Since $1 \neq 2$, the $\boxed{\lim_{(x,y)\to(0,0)} \frac{y}{x} \text{ cannot exist}}$.

(b) If $f(x,y,z) = e^{yz}\sin(xy)$, please compute f_{yz}.

1st w.r.t. y, 2nd w.r.t. z

$f(x,y,z) = \underbrace{e^{yz}}\underbrace{\sin(xy)}$ ← product rule for deriv with respect to y

$f_y(x,y,z) = ze^{yz}\sin(xy) + e^{yz}\cos(xy)x$

$= \underbrace{z}\underbrace{e^{yz}}\,\underbrace{\sin(xy)} + \underbrace{xe^{yz}}\cos(xy)$

product rule for deriv w.r.t. z

$f_{yz}(x,y,z) = \boxed{e^{yz}\sin(xy) + zye^{yz}\sin(xy) + xye^{yz}\cos(xy)}$

4. (a) Let $\bar{u} = (6, -2)$ and $\bar{v} = (2, 1)$. Compute $\text{proj}_{\bar{v}}\bar{u}$, the projection of $\bar{u}$ onto $\bar{v}$.

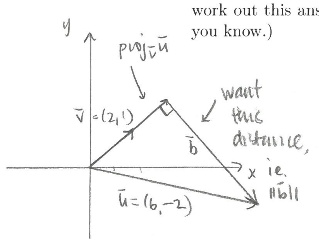
$$\text{proj}_{\bar{v}}\bar{u} = \frac{\bar{u}\cdot\bar{v}}{\bar{v}\cdot\bar{v}}\bar{v} = \frac{(6,-2)\cdot(2,1)}{(2,1)\cdot(2,1)}(2,1)$$

$$= \frac{10}{5}(2,1)$$

$$= \boxed{(4,2)}$$

(b) Use your answer to Part (a) to find the distance from the point $(6, -2)$ to the line through the origin with direction vector $(2, 1)$. Show all work and explain your reasoning.

(Note: you do not need to know a formula here. It is possible to work out this answer directly using the facts about vectors that you know.)



two possible methods:

① Notice $\text{proj}_{\bar{v}}\bar{u} + \bar{b} = \bar{u}$

So $\bar{b} = \bar{u} - \text{proj}_{\bar{v}}\bar{u}$

$= (6,-2) - (4,2)$

$= (2,-4)$

So Distance $= \|(2,-4)\| = \sqrt{4+16}\ \boxed{\sqrt{20}}$

② Pythagorean theorem:

$\|\bar{u}\|^2 = \|\text{proj}_{\bar{v}}\bar{u}\|^2 + \|\bar{b}\|^2$

$\|\bar{u}\|^2 = (\sqrt{36+4})^2 = (\sqrt{40})^2 = 40$

$\|\text{proj}_{\bar{v}}\bar{u}\|^2 = (\sqrt{16+4})^2 = (\sqrt{20})^2 = 20$

So $\|\bar{b}\|^2 = 40 - 20 = 20$

So $\boxed{\|\bar{b}\| = \sqrt{20}}$ (distance)

5. At what point on the graph of

$$f(x, y) = x^2 + 3y^2$$

is the tangent plane parallel to the plane $4x - 6y + z = 7$? Please show all work and explain your reasoning.

normal to this plane is $(4,6,1)$.

The normal to the tangent plane to the graph of f at $(x,y,f(x,y))$ is given by:

$(f_x(x,y), f_y(x,y), -1)$ (See your notes for why)

$= (2x, 6y, -1)$.

We want the values $(x,y,f(x,y))$ for which

$(2x, 6y, -1) \parallel (4,-6,1)$ parallel vectors are scalar multiples of each other

ie. $(2x, 6y, -1) = k(4,-6,1)$

From the third term, $k=-1$. So we want (x,y) s.t.

$2x = -4$

$6y = 6$

$\Rightarrow x = -2,\ y = 1.$

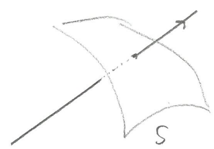
$(-2)^2 + 3(1)^2$

Thus the point where the tangent plane is parallel is $\boxed{(-2,1,7)}$

6. At what point(s) does the line

$$\bar{r}(t) = (t, 3+t, 3-t)$$

intersect the surface $xy + z = 2$? Show all work and explain your reasoning.



All points on the line equal

$(x, y, z) = (t, 3+t, 3-t)$

for some t.

All points (x, y, z) on the surface satisfy $xy + z = 2$.

Substitute (x, y, z) values of line into equation for surface to find the t for which the line intersects:

$(t)(3+t) + (3-t) = 2$

$\Rightarrow 3t + t^2 + 3 - t = 2$

$\Rightarrow t^2 + 2t + 1 = 0$

$\Rightarrow (t+1)(t+1) = 0 \Rightarrow t = -1.$

$(-1)(2) + 4 = 2$ ✓

On the line, this is the point

$\bar{r}(-1) = (-1, 3-1, 3-(-1)) = \boxed{(-1, 2, 4)}$

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (\sqrt{2}x - \sqrt{2}y, \sqrt{2}x + \sqrt{2}y).$$

(Note that f acts on $\mathbb{R}^2$ by a rotation of 45° followed by a dilation by $\sqrt{2}$.)

(a) Compute $Df(0, 1)$.

$$Df(0,1) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

(b) Compute $Df(0, 1)(\bar{e}_1)$ and $Df(0, 1)(\bar{e}_2)$, where $\bar{e}_1$ and $\bar{e}_2$ are the standard basis vectors. (In order to strengthen your intuition, before computing these, make a guess in your head what they should be and then see if you are right. You don't have to write your guess down.)

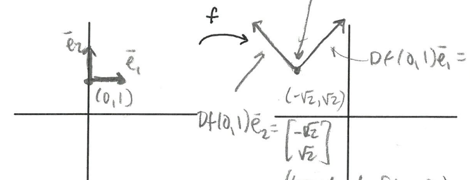
$Df(0,1)$ mimics f... since f is linear, $Df(0,1)$ will also rotate by 45° and dilate by $\sqrt{2}$.

$\bar{e}_1 = (1, 0)$ $\qquad$ $\bar{e}_2 = (0, 1)$

$$Df(0,1)\bar{e}_1 = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \qquad Df(0,1)\bar{e}_2 = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

(c) Place (and label) the vectors $Df(0, 1)(\bar{e}_1)$ and $Df(0, 1)(\bar{e}_2)$ in the appropriate position on the image below.

$f(0,1) = (-\sqrt{2}, \sqrt{2})$



(d) Use your answer to part (a) to estimate the value of $f(0.1, 1.1)$.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1.1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$f(0.1, 1.1) \approx f(0,1) + Df(0,1)\begin{bmatrix} 0.1 - 0 \\ 1.1 - 1 \end{bmatrix}$$

$f(0,1)$ written as a column vector

$$= \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} 0.1(\sqrt{2}) - 0.1(\sqrt{2}) \\ 0.1(\sqrt{2}) + 0.1(\sqrt{2}) \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ (1.2)(\sqrt{2}) \end{bmatrix}$$