Math 223 - Multivariable Calculus Name: solutions

Practice Exam 2

Please be sure to neatly **show and explain all of your work** and clearly label your answers. Other than your index card, this is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

Signed:

Problem	Points
1	/12
2	/10
3	/10
4	/12
5	/12
6	/12
7	/12
Total	/80

1. Please compute the following. Show all work.

(a) What is the arc length of the curve $\bar{r}(t) = (2t, t^2, \frac{1}{3}t^3)$, $0 \le t \le 1$?

$\bar{r}(t) = (2t, t^2, \frac{1}{3}t^3)$

$\bar{r}'(t) = (2, 2t, t^2)$

$\|\bar{r}'(t)\| = \sqrt{4 + 4t^2 + t^4}$

$= \sqrt{(2+t^2)^2}$ always positive

$= 2 + t^2.$

So arc length is:

$= \int_0^1 \|\bar{r}'(t)\| \, dt$

$= \int_0^1 2 + t^2 \, dt$

$= 2t + \frac{1}{3}t^3 \Big|_0^1$

$= [2 + \frac{1}{3}] - [0]$

$= \boxed{\frac{7}{3}}$

(b) If $\bar{F}(x,y,z) = (xz^2, x^2e^{yz}, x^2y^3z^4)$, please compute curl$\bar{F}$.

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & x^2e^{yz} & x^2y^3z^4 \end{vmatrix}$$

(like a cross product)

$$= \boxed{\left(3x^2y^2z^4 - x^2ze^{yz},\ 2xz - 2xy^3z^4,\ 2xe^{yz} - 0\right)}$$

2. Consider the function $f(x,y) = x^2 + 3xy$. Let $\bar{a} = (0,1)$.

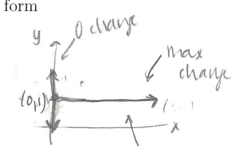
(a) Starting at the point $\bar{a}$, in what direction(s) is the directional derivative of f the greatest? Please give your answer in the form of (a) vector(s) and justify your response.

$\nabla f(\bar{a})$ points in direction of greatest directional derivative.

$\nabla f(x,y) = (2x+3y,\ 3x)$

$\nabla f(0,1) = (3,0)$ ← Starting from $\bar{a}$, move in this direction for greatest directional derivative.

(b) Starting at the point $\bar{a}$, in what direction(s) is the directional derivative of f equal to 0? Please give your answer in the form of (a) vector(s) and justify your response.



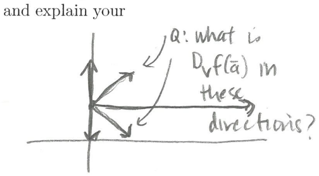
We want $\bar{v}$ so that $D_{\bar{v}} f(\bar{a}) = 0$.

from above → If $\bar{v}$ is a unit vector, $D_{\bar{v}} f(\bar{a}) = \nabla f(\bar{a}) \cdot \bar{v}$.

Here $\nabla f(\bar{a}) = (3,0)$.

Letting $\boxed{\bar{v} = (0,1) \text{ or } (0,-1)}$, we have $D_{\bar{v}} f(\bar{a}) = (3,0)\cdot(0,\pm 1) = 0$.

(c) What is the rate of change of f in the direction of a unit vector that points halfway between the direction of greatest change and the direction of 0 change? Please show all work and explain your reasoning.



unit vectors!

Here, $\bar{v} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

In this case, for either choice of $\bar{v}$,

$D_{\bar{v}} f(0,1) = \nabla f(0,1) \cdot \left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$= (3,0)\cdot\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$= \boxed{\frac{3}{\sqrt{2}}}$

3. Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is a scalar-valued function on $\mathbb{R}^3$ and suppose $\bar{r}: \mathbb{R} \to \mathbb{R}^3$ defines a curve in $\mathbb{R}^3$, with $\bar{r}(0) = \bar{a}$. Let the component functions of $\bar{r}(t)$ be $(x(t), y(t), z(t))$.

Since f is defined on all of $\mathbb{R}^3$, it makes sense to consider $f \circ \bar{r}$. (We often think of this as the restriction of f to the curve defined by $\bar{r}$.)

Notice that $f \circ \bar{r}: \mathbb{R} \to \mathbb{R}$.

Show that

$$\frac{d}{dt}(f \circ \bar{r})_{|t=0} = \nabla f(\bar{a}) \cdot \bar{r}'(0).$$

Be sure to justify each step that you make.

(Hint: as a first step, it is helpful to note that for any function $h: \mathbb{R} \to \mathbb{R}$, the notation $\frac{d}{dt}(h)_{|t=b}$ means the same thing as $Dh(b)$.)

(see notes about tangent plane to level surface)

Starting from the hint, we have

$\frac{d}{dt}(f\circ \bar{v})\big|_{t=0} = D(f\circ r)\big|_{t=0}$

Note: each evaluated at the point in their respective domains

chain rule!! (yay!) → $= Df(\bar{r}(0))\, D\bar{r}(0)$

$\bar{r}(0) = \bar{a}$ → $= Df(a)\, D\bar{r}(0)$

compute derivative matrices → $= \begin{bmatrix} f_x(a) & f_y(\bar{a}) & f_z(\bar{a}) \end{bmatrix} \begin{bmatrix} x'(0) \\ y'(0) \\ z'(0) \end{bmatrix}$

matrix mult → $= f_x(\bar{a})x'(0) + f_y(\bar{a})y'(0) + f_z(\bar{a})z'(0)$

same as vector dot product. → $= \nabla f(\bar{a}) \cdot \bar{r}'(0)$.

4. Consider $f(x, y) = 2x^2 + 3xy + 4x - 6y$

(a) Please find all critical points of $f(x, y)$. Show all work.

we need to find (x,y) s.t. $Df(x,y) = [0 \quad 0]$.

$Df(x,y) = [\underbrace{4x + 3y + 4}_{f_x} \quad \underbrace{3x - 6}_{f_y}]$

$f_y = 0$ when $3x - 6 = 0 \Rightarrow x = 2$.

when $x = 2$, $f_x(2,y) = 8 + 3y + 4 = 3y + 12 = 0$ when $y = -4$.

So we have one critical point: $\boxed{(2,-4)}$

(b) Choose a critical point from part (b) and classify it as a local maximum, local minimum, or neither.

to classify $(2,-4)$, we compute $Hf(2,-4)$:

$Hf(x,y) = \begin{bmatrix} 4 & 3 \\ 3 & 0 \end{bmatrix}$ So $Hf(2,-4) = \begin{bmatrix} 4 & 3 \\ 3 & 0 \end{bmatrix}$

$Hf(x,y)$ constant for all (x,y), including $(2,-4)$

By the second derivative test: $\det Hf(2,-4) = -9 < 0$,

So we have a $\boxed{\text{saddle point at } (2,-4)}$

5. Find the absolute minimum value of

$$f(x, y) = x^2 + 2y^2$$

on the circle $x^2 + y^2 = 1$. Show all work and clearly justify why you know you have found a maximum.

Use Lagrange multipliers on $f(x,y) = x^2 + 2y^2$

with constraint $\underbrace{x^2 + y^2}_{g(x,y)} = \underbrace{1}_{c}$

We have $\nabla f = (2x, 4y)$

$\nabla g = (2x, 2y)$

So $\nabla f = \lambda \nabla g$ gives $\begin{cases} 2x = \lambda 2x & (1) \\ 4y = \lambda 2y & (2) \end{cases}$

and constraint : $x^2 + y^2 = 1$ (3)

From (1), $2x - \lambda 2x = 0 \Rightarrow 2x(1-\lambda) = 0 \Rightarrow x = 0$ or $\lambda = 1$

- if $x = 0$, by (3), $y = \pm 1$ $\rightsquigarrow$ consider $(0,1), (0,-1)$
- if $\lambda = 1$, by (2) $4y = 2y \Rightarrow y = 0$. Then by (3), $x = \pm 1$ $\rightsquigarrow$ consider $(1,0), (-1,0)$

By $\boxed{\text{Extreme Value Thm}}$, since the circle $x^2+y^2=1$ is closed and bounded, f must achieve an abs min on the circle. Thus, test points and choose smallest

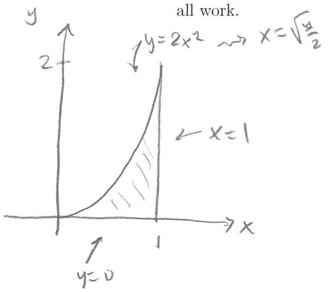
$f(x,y) = x^2 + 2y^2$

$f(0, \pm 1) = 2$

$f(\pm 1, 0) = 1$

$\boxed{\text{abs. min value is 1}}$

6. Compute $\iint_R f(x,y)\,dA$ where $f(x,y) = x^2y$ and R is the region in the xy-plane bounded by the curves $y=0$, $x=1$, and $y=2x^2$. Show all work.



$$\int_0^1\int_0^{2x^2} x^2y\,dy\,dx$$

$$= \int_0^1 \frac{x^2y^2}{2}\Big|_0^{2x^2} dx$$

$$= \int_0^1 \frac{x^2(4x^4)}{2} - 0\,dx$$

$$= \int_0^1 2x^6\,dx$$

$$= \frac{2}{7}x^7\Big|_0^1 = \boxed{\frac{2}{7}}$$

easier!

Or!

$$\int_0^2\int_{\sqrt{\frac{y}{2}}}^1 x^2y\,dx\,dy$$

$$= \int_0^2 \frac{x^3y}{3}\Big|_{\sqrt{\frac{y}{2}}}^1 dy$$

$$= \int_0^2 \frac{y}{3} - \frac{y^{5/2}}{6\sqrt{2}}\,dy$$

$$= \frac{y^2}{6} - \frac{2y^{7/2}}{42\sqrt{2}}\Big|_0^2$$

$$= \left[\frac{4}{6} - \frac{2(8)\sqrt{2}}{42\sqrt{2}}\right] - [0]$$

$$= \frac{2}{3} - \frac{8}{21}$$

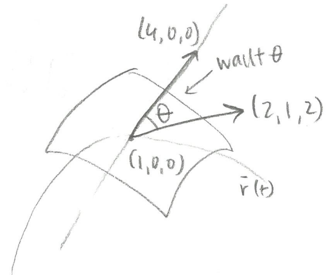
$$= \frac{14-8}{21} = \frac{6}{21} = \boxed{\frac{2}{7}}$$

7. Observe that the point $(1,0,0)$ lies on both the surface $2x^2+y^2+z^2=2$ and the curve $\bar{r}(t) = (t^2, t-1, 2t-2)$. (How do you know?)

At the point $(1,0,0)$, at what angle does the tangent line to $\bar{r}(t)$ intersect the normal line to the surface? Show all work and explain your reasoning.

Since $2(1)^2+0^2+0^2=2$, the pt lies on the surface.

Since $\bar{r}(1) = (1^2, 1-1, 2(1)-2) = (1,0,0)$, the point lies on the curve at $t=1$.



Direction vector of normal line to surface is

$$\nabla f(1,0,0)$$

$$\nabla f(x,y,z) = (4x, 2y, 2z)$$

$$\nabla f(1,0,0) = (4,0,0)$$

Direction vector of line is $\bar{r}'(1)$.

$$\bar{r}'(t) = (2t, 1, 2)$$

$$\bar{r}'(1) = (2,1,2)$$

$$(4,0,0)\cdot(2,1,2) = \|(4,0,0)\|\|(2,1,2)\|\cos\theta$$

$$8 = 4\sqrt{4+1+4}\cos\theta \rightsquigarrow \frac{8}{12} = \cos\theta \rightsquigarrow \boxed{\theta = \cos^{-1}\left(\frac{2}{3}\right)}$$