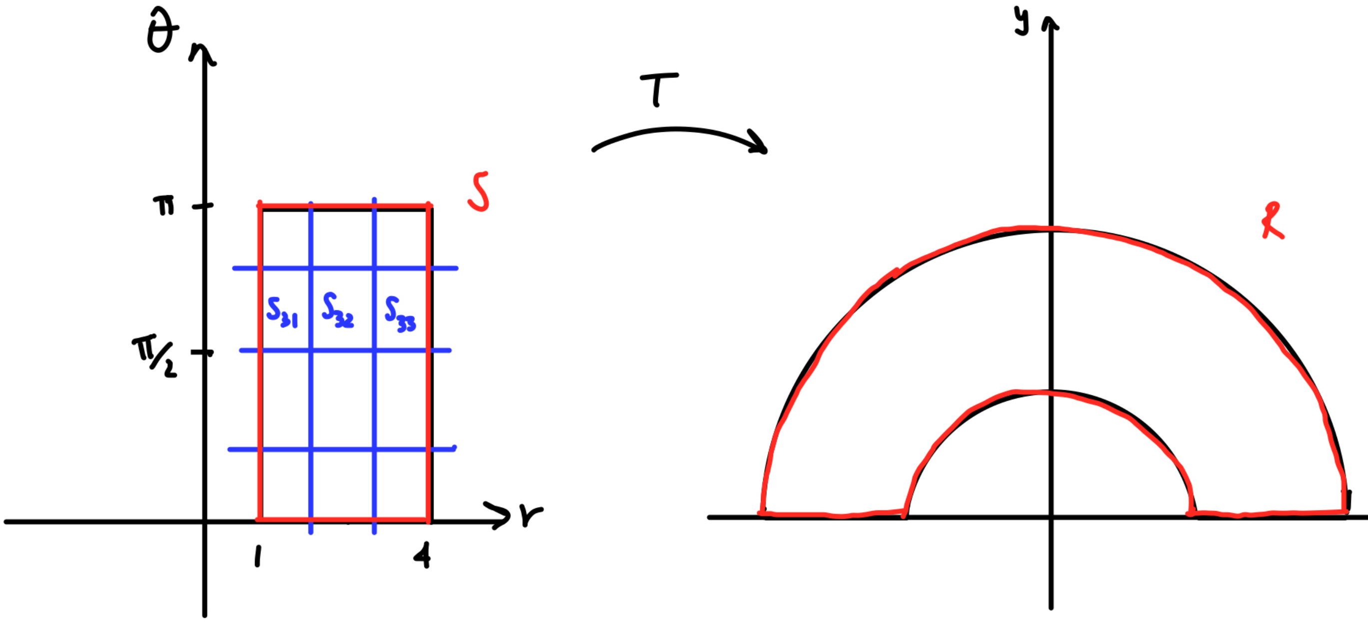
Math 223: Multivariable Calculus
Change of Variables in Integration

Consider the polar coordinate transformation $T(r, \theta) = (r\cos\theta, r\sin\theta)$. In this worksheet, we will trace the progression of the notation/theory behind the change of variables theorem for the special case of the polar coordinate transformation, evaluated at particular points.



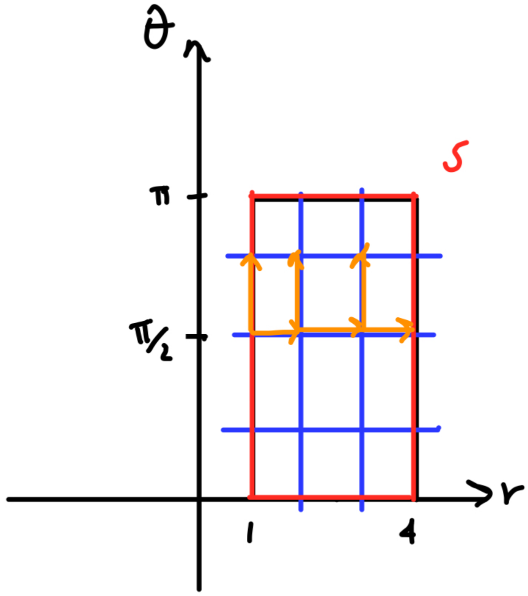
1. The region $S = [1,4] \times [0, \pi]$ in the $r\theta$-plane above has been partitioned into subregions S_{ij}. Draw the corresponding partition of R that arises from applying the coordinate transformation T to the partition of S.

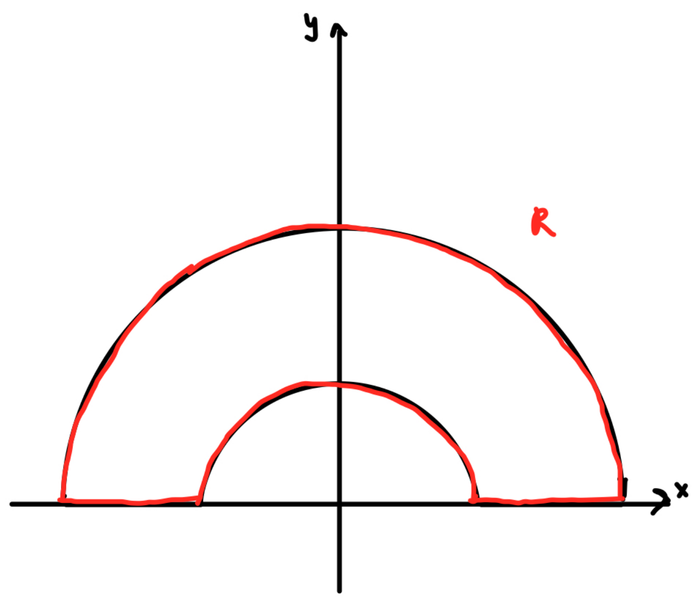
2. The subregions S_{31}, S_{32}, S_{33} are labelled on S. Label the corresponding subregions R_{31}, R_{32}, R_{33} on R.

3. Just for a reality check/intuition: are the areas of S_{31}, S_{32}, S_{33} equal to each other? Are the areas of R_{31}, R_{32}, R_{33} equal to each other? What simple conclusions can you draw from this about T and distortion?

The questions above referred to the polar coordinate transformation T, mapping the region S to the region R. We now turn our attention to the *derivative* DT of T, computed at three representative points in S.

Recall that $DT(r_0, \theta_0)$ maps vectors based at (r_0, θ_0) to vectors based at $T(r_0, \theta_0)$.





4. For each point (r_0, θ_0) equal to $(1, \frac{\pi}{2})$, $(2, \frac{\pi}{2})$, and $(3, \frac{\pi}{2})$, compute $DT(r_0, \theta_0)$.

5. The basis $\{\bar{e}_1, \frac{\pi}{4}\bar{e}_2\}$ has been drawn at each of the base points $(1, \frac{\pi}{2}), (2, \frac{\pi}{2}), (3, \frac{\pi}{2})$ of S_{31}, S_{32}, S_{33} respectively. For each of these points (r_0, θ_0), compute

$$DT(r_0, \theta_0)(\bar{e}_1) \quad \text{and} \quad DT(r_0, \theta_0)(\tfrac{\pi}{4}\bar{e}_2).$$

6. Plot your answers to Question 5 on the picture of R above. Let P_{31}, P_{32}, P_{33} denote the parallelograms defined by your answers to Question 5.

7. For each of the three points $(1, \frac{\pi}{2})$, $(2, \frac{\pi}{2})$, and $(3, \frac{\pi}{2})$, compute $\det DT(r_0, \theta_0)$.

8. What are the areas of P_{31}, P_{32}, P_{33}?

9. Now, for a general point (r, θ) in S, if vectors $(\Delta r)\bar{e}_1$ and $(\Delta\theta)\bar{e}_2$ define a tangent parallelogram Q based at (r, θ), what is the area of the corresponding tangent parallelogram $P = DT(r, \theta)(Q)$ based at $T(r, \theta)$? How is the area of P related to the area of Q?

Your answer to Question 9 above tells us what the distortion factor should be when pulling an integral in the xy-plane back to the the $r\theta$-plane via the polar coordinate transformation. Namely, if R is a region in the xy-plane that corresponds to region S in the $r\theta$-plane, then

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta) r\, dr\, d\theta. \tag{1}$$

10. Consider the integral $\iint_R xy\, dA$ over the region R in the xy-plane given above. If you were to compute this overall integral using rectangular coordinates, how many integrals would you need to set up? (You don't need to set them up!)

11. Try using the Formula (1) above to compute the value of this integral.