The Second Derivative Test

(for $f : \mathbb{R}^2 \to \mathbb{R}$)

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2. Taylor's theorem says that for $\bar{x}$ near $\bar{a}$,

$$f(\bar{x}) \approx f(\bar{a}) + Df(\bar{a})(\bar{x} - \bar{a}) + \frac{1}{2}(\bar{x} - \bar{a})^{\mathrm{T}} Hf(\bar{a})(\bar{x} - \bar{a})$$

where

$$Hf(\bar{a}) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

If $\bar{a}$ is a critical point, then $Df(\bar{a}) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ so

$$f(\bar{x}) \approx f(\bar{a}) + \frac{1}{2}(\bar{x} - \bar{a})^{\mathrm{T}} Hf(\bar{a})(\bar{x} - \bar{a}).$$

By choosing $\bar{x}$ close enough to $\bar{a}$, we can make the difference between these actual values as small as we like.

Thus near $\bar{a}$, the difference between $f(\bar{x})$ and $f(\bar{a})$ is determined by the sign of

$$(\bar{x} - \bar{a})^{\mathrm{T}} Hf(\bar{a})(\bar{x} - \bar{a})$$

Note that $\bar{x} - \bar{a}$ can be thought of as a vector based at $\bar{a}$.

Letting $\bar{h} = \bar{x} - \bar{a}$, we have

$$f(\bar{x}) \approx f(\bar{a}) + \frac{1}{2} \underbrace{\bar{h}^{\mathrm{T}} H f(\bar{a}) \bar{h}}_{\circledast}.$$

Thus, f has:

- a local minimum at $\bar{a}$ if $\circledast > 0$ for all $\bar{h} \neq \bar{0}$,
- a local maximum at $\bar{a}$ if $\circledast < 0$ for all $\bar{h} \neq \bar{0}$,
- a saddle point at $\bar{a}$ if $\circledast > 0$ for some $\bar{h}$ and $\circledast < 0$ for other $\bar{h}$.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 and $Df(\bar{a}) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

The second derivative test says that

- f has a local minimum at $\bar{a}$ if $\det Hf(\bar{a}) > 0$ and $f_{xx} > 0$,

  (this will imply $\circledast > 0$ for all $\bar{h} \neq \bar{0}$)

- f has a local maximum at $\bar{a}$ if $\det Hf(\bar{a}) > 0$ and $f_{xx} < 0$,

  (this will imply $\circledast < 0$ for all $\bar{h} \neq \bar{0}$)

- f has a saddle point at $\bar{a}$ if $\det Hf(\bar{a}) < 0$.

  (this will imply $\circledast > 0$ for some $\bar{h}$ and $\circledast < 0$ for other $\bar{h}$)

Why?

Since $Hf(\bar{a})$ is symmetric, it is diagonalizable.

Thus, there is an invertible matrix P such that

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = P^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P.$$

where λ_1 and λ_2 are the eigenvalues of $Hf(\bar{a})$.

It turns out that P can be chosen so that $P^{-1} = P^{\mathrm{T}}$. (P is called an orthogonal matrix.)

Let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \bar{v} = P\bar{h}$. Then

$$\begin{aligned}
\circledast &= \bar{h}^{\mathrm{T}} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \bar{h} \\
&= \bar{h}^{T} P^{\mathrm{T}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P\bar{h} \\
&= (P\bar{h})^{\mathrm{T}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P\bar{h} \\
&= \bar{v}^{\mathrm{T}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \bar{v} \\
&= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= \lambda_1 v_1^2 + \lambda_2 v_2^2.
\end{aligned}$$

Since $\circledast = \lambda_1 v_1^2 + \lambda_2 v_2^2$, we conclude

- $\circledast > 0$ for all $\bar{h} \neq \bar{0}$ if λ_1 and λ_2 are both positive,
- $\circledast < 0$ for all $\bar{h} \neq \bar{0}$ if λ_1 and λ_2 are both negative,
- $\circledast$ is mixed if λ_1 and λ_2 have opposite signs.

Similar matrices have the same determinant and the same trace.

Thus,

$$f_{xx}f_{yy} - (f_{xy})^2 = \det Hf(\bar{a}) = \det \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1\lambda_2$$

and

$$f_{xx} + f_{yy} = \operatorname{trace} Hf(\bar{a}) = \operatorname{trace} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1 + \lambda_2.$$

Recall: the second derivative test says that if $Df(\bar{a}) = \begin{bmatrix} 0 & 0 \end{bmatrix}$

- f has a local minimum at $\bar{a}$ if $\det Hf(\bar{a}) > 0$ and $f_{xx} > 0$,

  (this will imply $\circledast > 0$ for all $\bar{h} \neq \bar{0}$)

- f has a local maximum at $\bar{a}$ if $\det Hf(\bar{a}) > 0$ and $f_{xx} < 0$,

  (this will imply $\circledast < 0$ for all $\bar{h} \neq \bar{0}$)

- f has a saddle point at $\bar{a}$ if $\det Hf(\bar{a}) < 0$.

  (this will imply $\circledast > 0$ for some $\bar{h}$ and $\circledast < 0$ for other $\bar{h}$)

Suppose $\det Hf(\bar{a}) < 0$.

Then $\lambda_1 \lambda_2 < 0$, so λ_1 and λ_2 have opposite signs.

By above, this means $\circledast > 0$ for some $\bar{h}$ and $\circledast < 0$ for other $\bar{h}$,

so f has a saddle point at $\bar{a}$. ✓

Now, suppose $\det Hf(\bar{a}) > 0$.

Then $\lambda_1 \lambda_2 > 0$ so λ_1 and λ_2 have the same sign.

Also, $f_{xx} f_{yy} - (f_{xy})^2 > 0$, so $f_{xx} f_{yy} > 0$.

Thus f_{xx} and f_{yy} have the same sign as well.

But $f_{xx} + f_{yy} = \lambda_1 + \lambda_2$.

Thus $f_{xx}, f_{yy}, \lambda_1$, and λ_2 *all* have the same sign.

Therefore, if $\det Hf(\bar{a}) > 0$ and $f_{xx} > 0$,

λ_1 and λ_2 are both positive, so by above $\circledast > 0$ for all $\bar{h}$,

and f has a local minimum at $\bar{a}$. ✓

And if $\det Hf(\bar{a}) > 0$ and $f_{xx} < 0$,

λ_1 and λ_2 are both negative, so by above $\circledast < 0$ for all $\bar{h}$,

and f has a local maximum at $\bar{a}$. ✓

Note:

If $\det Hf(\bar{a}) = 0$, the second derivative test is inconclusive.

Could be that f has a local minimum, local maximum, or saddle point at $\bar{a}$.

Need to make some other argument based on f to classify the critical point.