Math 223: Multivariable Calculus
Example of the Second Derivative Test

Sometimes the algebra of finding critical points can be a little messy. Here is a simple example that highlights a useful trick to keep in mind.

Example 0.1. Suppose $f(x, y) = x^2y - 4xy + y^2$. Find the critical points of $f(x, y)$.

We must find all points $\bar{a}$ such that $Df(\bar{a}) = \begin{bmatrix} 0 & 0 \end{bmatrix}$. This amounts to finding all points $\bar{a}$ such that $f_x(\bar{a}) = f_y(\bar{a}) = 0$. We compute

$$\begin{aligned} f_x(x, y) &= 2xy - 4y = 2y(x - 2) \\ f_y(x, y) &= x^2 - 4x + 2y \end{aligned}$$

Here is a useful trick/technique for solving a system like this: Notice that f_x in this case can be written as a simple product and observe that the only way that the product can equal 0 is if one or the other of the factors equals 0.

Thus $f_x(x, y) = 0$ when $y = 0$ or when $x = 2$. This allows us to break into two cases.

Case 1: $y = 0$. In this case, $f_y(x, y) = x^2 - 4x + 2(0) = x^2 - 4x = x(x - 4)$. So $f_y(x, y)$ will also equal 0 if $x = 0$ or $x = 4$. Thus we have two critical points: $(0, 0)$ and $(4, 0)$.

Case 2: $x = 2$. In this case, $f_y(x, y) = (2)^2 - 4(2) + 2y = -4 + 2y$. So $f_y(x, y)$ will also equal 0 if $y = 2$. Thus we have a third critical point: $(2, 2)$.

Now that we have the critical points, we can classify each one. We'll just do one here:

Example 0.2. Classify the critical point $(2, 2)$ of $f(x, y) = x^2y - 4xy + y^2$.

The Hessian of f is given by

$$Hf(x, y) = \begin{bmatrix} 2y & 2x - 4 \\ 2x - 4 & 2 \end{bmatrix}.$$

At $\bar{a} = (2, 2)$, this is

$$Hf(2, 2) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

Thus $\det Hf(2, 2) = 8 > 0$. Furthermore, $f_{xx}(2, 2) = 4 > 0$. Therefore, the second derivative test tells us that f has a local minimum at $(2, 2)$.