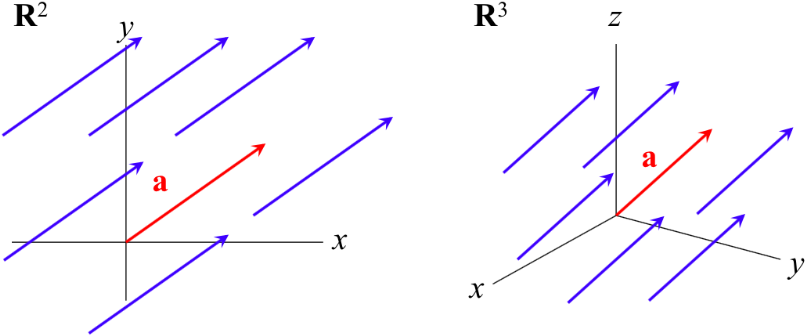
Sections 1.1 and 1.2: Vectors in Two and Three Dimensions

Algebraic and Geometric Points of View

Unlike points, we think of vectors as mobile in space.

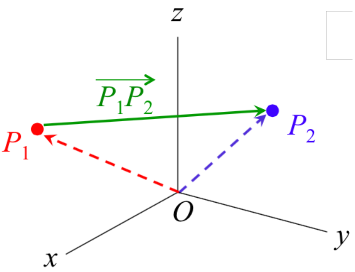


If $\overline{a} = (a_1, a_2, a_3)$, then the red vector represents the position vector of the *point* $P = (a_1, a_2, a_3)$.

The displacement vector between two points is given by the difference of their position vectors.

$$\text{If } P_1 = (x_1, y_1, z_1) \text{ and } P_2 = (x_2, y_2, z_2) \text{ then}$$

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$



Parametric Equations of Lines

To describe a line (or any curve) in $\mathbb{R}^n$, use vector or parametric equations with one parameter, t.

For a line, the essential pieces of information you need are:

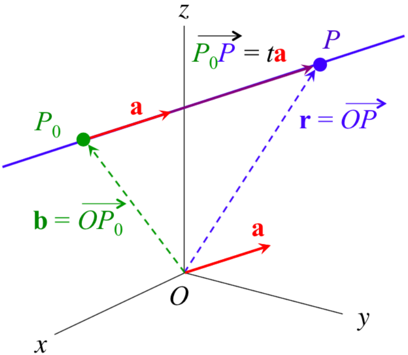
- a point
- a direction vector

Given

- a specific point $P = (b_1, b_2, b_3)$ on the line and
- a vector $\overline{a} = (a_1, a_2, a_3)$ parallel to the line,

the position vector of a general point $\overline{r}$ on the line can be written

$$\overline{r}(t) = \overline{b} + t\overline{a}, \quad t \in \mathbb{R}$$



$$\overline{r}(t) = \overline{b} + t\overline{a}, \quad t \in \mathbb{R}.$$

- The expression above is called the vector equation of a line.

- The vectors $\overline{b}$ and $\overline{a}$ are *fixed*. The scalar t is *variable*, and is referred to as a parameter.

- In terms of components,

$$\overline{r}(t) = (x(t), y(t), z(t)) = (b_1 + ta_1, b_2 + ta_2, b_3 + ta_3)$$

which gives us the parametric equations of a line:

$$\begin{aligned} x(t) &= b_1 + ta_1 \\ y(t) &= b_2 + ta_2 \\ z(t) &= b_3 + ta_3. \end{aligned}$$

Example

Line between $(3, 1, 4)$ and $(2, 3, 7)$.

Displacement vector:

$$(2, 3, 7) - (3, 1, 4) = (2 - 3, 3 - 1, 7 - 4) = (-1, 2, 3)$$

Point: $(3, 1, 4)$

Vector equation of line:

$$(3, 1, 4) + t(-1, 2, 3), t \in \mathbb{R}.$$

Parametric equations of line:

$$\begin{aligned} x(t) &= 3 - t \\ y(t) &= 1 + 2t \\ z(t) &= 4 + 3t \end{aligned}$$

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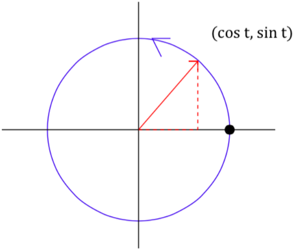
Parametric Equations of Curves

More generally, a set of parametric equations in $\mathbb{R}^n$ determines a curve in $\mathbb{R}^n$.

For example,

$$x(t) = \cos t$$
$$y(t) = \sin t, \qquad 0 \le t \le 2\pi$$

defines a circle in $\mathbb{R}^2$.



Important observation:

- The parametric equations for a line or curve are not unique.
- There is always more than one way to describe a particular curve using parametric equations.