Sections 1.5 and 1.6: Equations of Planes and Some n-Dimensional Geometry

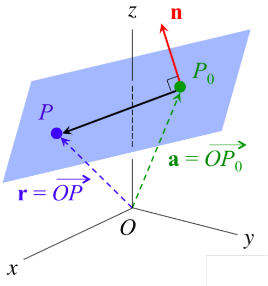
The Coordinate Equation of a Plane

A plane in $\mathbb{R}^3$ is determined by:

- a point P_0 on the plane
- and a normal vector $\bar{n}$ to the plane.

Suppose that

- $P_0 = (x_0, y_0, z_0)$ is a specific, *fixed* point in the plane,
- $P = (x, y, z)$ is an arbitrary, *variable* point on the plane,
- $\bar{n} = (A, B, C)$ is a normal vector to the plane.



For every point P in the plane, we have

$$\bar{n} \cdot \overrightarrow{P_0P} = 0.$$

The vector equation

$$\bar{n} \cdot \overrightarrow{P_0P} = 0$$

leads us to the coordinate equation of a plane:

$$\bar{n} = (A, B, C) \qquad \overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

so

$$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

implies

$$Ax + By + Cz = D$$

(where $D = Ax_0 + By_0 + Cz_0$).

Example

Find the coordinate equation of the plane with normal vector $\bar{n} = (-1, 5, 3)$ through the point $(2, 7, 3)$.

$$(-1, 5, 3) \cdot (x - 2, y - 7, z - 3) = 0$$

$$\implies -1(x - 2) + 5(y - 7) + 3(z - 3) = 0$$

$$\implies -x + 5y + 3z = 42.$$

Important observation: The coefficients of x, y, and z in a coordinate equation for a plane give a normal vector to the plane.

How would you find the equation for a plane:

- given 3 noncollinear points on the plane?

- given a line and a point in the plane?

We say that two planes are parallel if their normal vectors are parallel

> (in other words, if their normal vectors are nonzero scalar multiples of each other.)

The angle between two planes is the angle between their normal vectors.

Example

Find the vector equation of the line of intersection between the planes

$$\begin{aligned} 5x - y - 2z &= -1 \\ -x + y - 4z &= -3. \end{aligned}$$

Planes:

$$5x - y - 2z = -1$$
$$-x + y - 4z = -3.$$

$$\bar{n}_1 = (5, -1, -2)$$
$$\bar{n}_2 = (-1, 1, -4)$$

Direction vector for the line is: $\bar{n}_1 \times \bar{n}_2 = (6, 22, 4)$.

(Based on the observation that the line lies *in* both planes.)

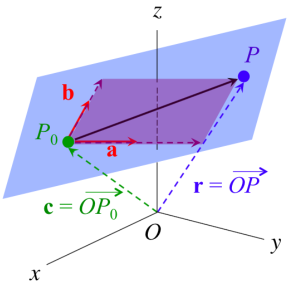
For a point on the plane, set $z = 0$, solve for (x, y): $(-1, -4, 0)$.

Vector equation for line of intersection: $(-1, -4, 0) + t(6, 22, 4)$.

Parametric Equations of a Plane

We can also define a plane parametrically if we have

- two vectors $\bar{a}$ and $\bar{b}$ parallel to the plane and
- a point P_0 in the plane with position vector $\bar{c}$.



The position vector of an arbitrary point $P = (x, y, z)$ in the plane is given by

$$\bar{c} + s\bar{a} + t\bar{b}, \quad s \in \mathbb{R}, \quad t \in \mathbb{R}.$$

(See the vector equation for a line.)

The distance between a point/line/plane and another point/line/plane is the shortest possible distance when varying over all points in both.

See Examples in Section 1.5. Good practice with vectors.

Section 1.6 has information about $\mathbb{R}^n$ and about matrices.

We will cover a two important inequalities. Read the rest for practice and/or review.

The Cauchy-Schwarz Inequality

An inequality based on the dot product.

Suppose that

$$\bar{x}, \bar{y} \in \mathbb{R}^n.$$

The Cauchy-Schwarz inequality says:

$$|\bar{x} \cdot \bar{y}| \leq \|\bar{x}\| \|\bar{y}\|.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this follows from

$$\bar{x} \cdot \bar{y} = \|\bar{x}\| \|\bar{y}\| \cos\theta.$$

In $\mathbb{R}^n$ for $n \geq 4$, a proof would be required.

Proof of the Cauchy-Schwarz Inequality

Let $\bar{x}, \bar{y} \in \mathbb{R}^n$. Suppose

$$\bar{z} = a\bar{x} + b\bar{y} \quad \text{for some } a \text{ and } b.$$

By properties of the dot product

$$\begin{aligned} 0 &\leq \bar{z} \cdot \bar{z} \\ &= (a\bar{x} + b\bar{y}) \cdot (a\bar{x} + b\bar{y}) \\ &= a^2\|\bar{x}\|^2 + 2ab(\bar{x} \cdot \bar{y}) + b^2\|\bar{y}\|^2. \end{aligned}$$

Let $a = \|\bar{y}\|^2$ and $b = -(\bar{x} \cdot \bar{y})$. Then our equation becomes:

$$0 \leq \|\bar{y}\|^4\|\bar{x}\|^2 - 2\|\bar{y}\|^2(\bar{x} \cdot \bar{y})^2 + (\bar{x} \cdot \bar{y})^2\|\bar{y}\|^2.$$

So far we have:

$$\begin{aligned} 0 &\leq \|\bar{y}\|^4\|\bar{x}\|^2 - 2\|\bar{y}\|^2(\bar{x}\cdot\bar{y})^2 + (\bar{x}\cdot\bar{y})^2\|\bar{y}\|^2 \\ &= \|\bar{y}\|^2(\|\bar{y}\|^2\|\bar{x}\|^2 - (\bar{x}\cdot\bar{y})^2). \end{aligned}$$

Clean up:

$$(\bar{x}\cdot\bar{y})^2 \leq \|\bar{x}\|^2\|\bar{y}\|^2$$

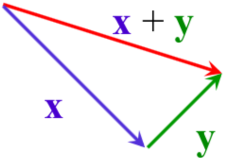
and take square roots:

$$|\bar{x}\cdot\bar{y}| \leq \|\bar{x}\|\|\bar{y}\|.$$

Done!

The Triangle Inequality

Suppose that $\bar{x}, \bar{y} \in \mathbb{R}^n$.



The triangle inequality says:

$$\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|.$$

Proof of the Triangle Inequality

The proof follows from the Cauchy-Schwarz inequality.

$$\begin{aligned}
\|\bar{x} + \bar{y}\|^2 &= (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) \\
&= \|\bar{x}\|^2 + 2\bar{x} \cdot \bar{y} + \|\bar{y}\|^2 \\
&\leq \|\bar{x}\|^2 + 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2 \quad \text{(by Cauchy-Schwarz)} \\
&= (\|\bar{x}\|^2 + \|\bar{y}\|^2)^2.
\end{aligned}$$

So $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$, as desired.