The Total Derivative

Goal: Generalize defn of differentiability of $f: \mathbb{R}^2 \to \mathbb{R}$ to a defn of differentiability for $f: \mathbb{R}^n \to \mathbb{R}^m$.

Recall: $f: \mathbb{R}^2 \to \mathbb{R}$ diff'ble at (a,b) if

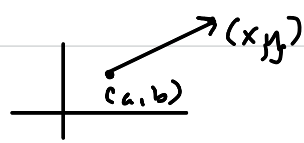
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - h(x,y)}{\|(x,y)-(a,b)\|} = 0.$$

where $h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Reexpress:

matrix mult

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - \overbrace{\begin{bmatrix} f_x(a,b) & f_y(a,b) \end{bmatrix}}^{\text{total derivative of } f,\ Df(a,b)} \begin{bmatrix} x-a \\ y-b \end{bmatrix}}{\|(x,y)-(a,b)\|} = 0.$$



Most generally: If $f: \mathbb{R}^n \to \mathbb{R}^m$, the <u>total derivative</u> of f at $\bar{a}$ is a linear transformation, denoted $Df(\bar{a})$, $Df(\bar{a}): \mathbb{R}^n \to \mathbb{R}^m$ which is an approximation of f near $\bar{a}$.

<u>Defn.</u> Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is differentiable at $\bar{a} \in \mathbb{R}^n$ if there exists a linear transformation called $Df(\bar{a}): \mathbb{R}^n \to \mathbb{R}^m$ such that if

$$h(\bar{x}) = f(\bar{a}) + [Df(\bar{a})](\bar{x}-\bar{a})$$

then

$$\lim_{\bar{x} \to \bar{a}} \frac{|f(\bar{x}) - h(\bar{x})|}{|\bar{x} - \bar{a}|} = 0.$$