Turns out: if $f = (f_1, \dots, f_m)$ $: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, the standard matrix of $Df(\bar{a})$ is:

$$Df(\bar{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{a}) \\ \frac{\partial f_2}{\partial x_1}(\bar{a}) & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\bar{a}) & & \frac{\partial f_m}{\partial x_n}(\bar{a}) \end{bmatrix}$$

<u>Ex</u> $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(r, \theta) = (r\cos\theta, r\sin\theta)$

analogy:

$$Df(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$f(x) = x^2$

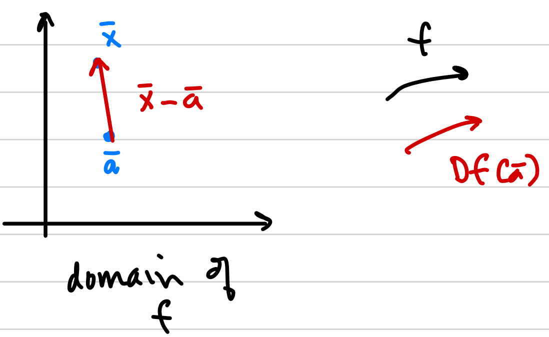
$f'(x) = 2x$

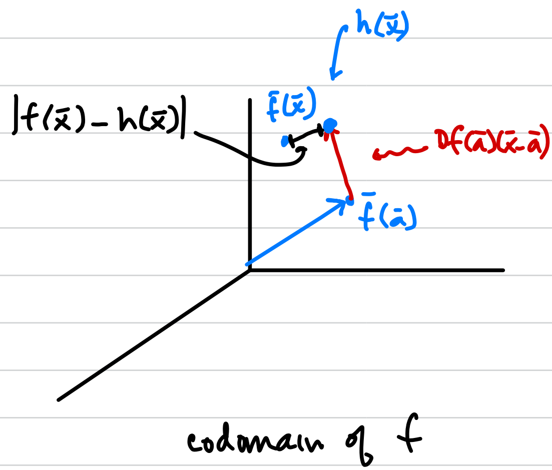
$$Df\left(2, \tfrac{\pi}{2}\right) = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$f'(2) = 4$

What is $Df(\bar{a})$ a linear transformation $\underline{\underline{\text{of}}}$?

Idea: $Df(\bar{a})$ maps vectors based at $\bar{a}$ to vectors based at $f(\bar{a})$.





bus stop

$$\bar{h}(\bar{x}) = f(\bar{a}) + Df(\bar{a})(\bar{x}-\bar{a})$$

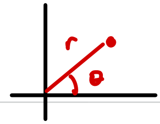
f diff'ble at $\bar{a}$ says $\bar{h}(\bar{x})$ is a good approximation of $f(\bar{x})$ near $\bar{a}$.

$Df(\bar{a})$ is a linear transformation

"differentiation is linearization."

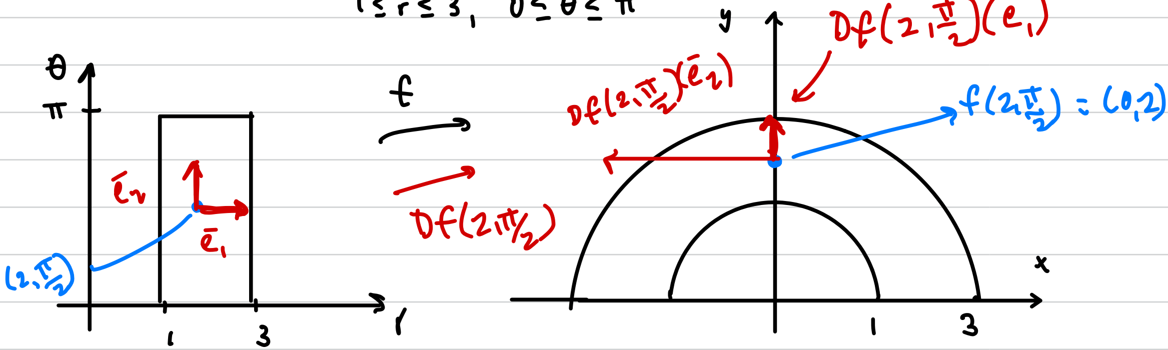
$x = r\cos\theta$

$y = r\sin\theta$



Ex $f(r, \theta) = (r\cos\theta, r\sin\theta)$

$1 \le r \le 3, \quad 0 \le \theta \le \pi$



$\bar{e}_1$

$$Df\left(2, \tfrac{\pi}{2}\right)(\bar{e}_1) = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\bar{e}_2$

$$Df\left(2, \tfrac{\pi}{2}\right)(\bar{e}_2) = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Try with $\bar{a} = \left(1, \tfrac{\pi}{2}\right)$, $\left(3, \tfrac{\pi}{2}\right)$, $\left(2, \tfrac{\pi}{4}\right)$, ...

Idea: Df "mimics" action of f near $\bar{a}$.