Tangent Planes and Level Surfaces

Recall: level surfaces

Sps $f : \mathbb{R}^3 \to \mathbb{R}$

i.e. output value

Level surface of f at level k is the set of all (x,y,z) satisfying:

$$f(x,y,z) = k.$$

e.g.

$$\begin{cases} 3x^2+y^2+4z^2 = 1 \\ 3x^2+y^2+4z^2 = 4 \\ 3x^2+y^2+4z^2 = 9 \end{cases}$$

<u>Ex.</u> $f(x,y,z) = 3x^2+y^2+4z^2$

↳ level surfaces are all: ellipsoids

<u>Ex.</u> $f : \mathbb{R}^2 \to \mathbb{R}$. Graph of f is 0-level surface of

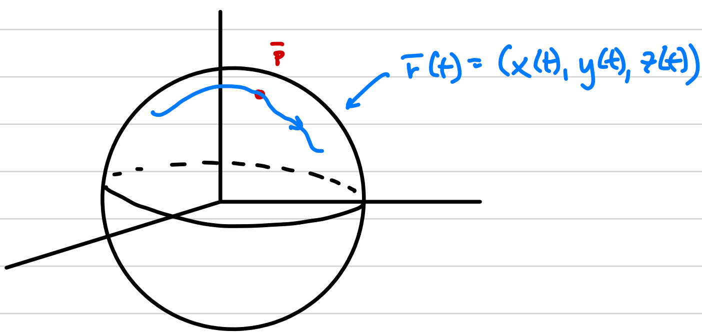
$z = f(x,y)$

$$F(x,y,z) = f(x,y) - z$$

Goal: tangent plane to level surface S at a point $\bar{p}$ in S.

Sps S is a level surface of $F(x,y,z)$.

$$S = \{ (x,y,z) \mid F(x,y,z) = k \}$$

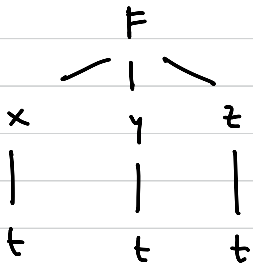


Key idea: Any curve $\bar{r}(t) = (x(t), y(t), z(t))$ on S through p satisfies:

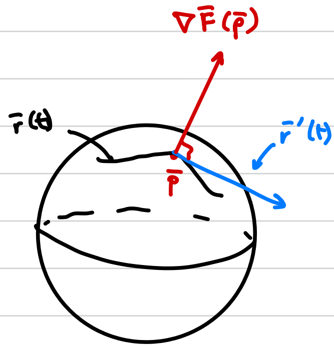
$$F(\bar{r}(t)) = F(x(t), y(t), z(t)) = k.$$

Differentiate both sides of ●.

But then, by chain rule:



$$\underbrace{\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}}_{\frac{d}{dt}\left(F(x(t),\, y(t),\, z(t))\right)} = \underbrace{0}_{\frac{d}{dt}(k)}.$$



i.e. $\nabla F(p) \cdot \bar{r}\,'(t) = 0$ so: $\nabla F(p) \perp \bar{r}\,'(t)$

generic tangent vector to S at $\bar{p}$.
i.e. arbitrary vector in tangent plane.

This is true for all curves $\bar{r}(t)$ on S, thus $\nabla F(\bar{p})$ is orthogonal to all tangent vectors to S at $\bar{p}$.

Conclusion: $\nabla F(\bar{p})$ is normal to tangent plane to S at $\bar{p}$.

Thus: The equation for the tangent plane to the level surface of $F(x,y,z)$ at $P=(x_0,y_0,z_0)$ is:

$$\underbrace{\nabla F(p)}_{\bar{n}} \cdot (\underbrace{\bar{r}}_{\text{general pt. } (x,y,z) \text{ in plane}} - \underbrace{\bar{r}_0}_{\text{specific point } (x_0,y_0,z_0) \text{ in plane}}) = 0$$

plane:
point: P
$\bar{n} = \nabla F(p)$.

ie.

$$(F_x(x_0,y_0,z_0), F_y(x_0,y_0,z_0), F_z(x_0,y_0,z_0)) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.}$$