Taylor Polynomials

Recall: Taylor's theorem for $f: \mathbb{R} \to \mathbb{R}$:

$p_k(x)$

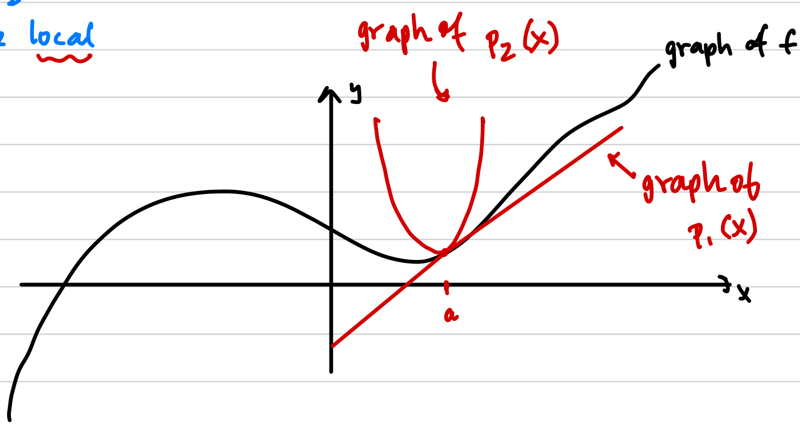
Near a,

tangent line ... recall defn of $Df(\bar{a})$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

$p_2(x)$ ... focus here now.

can use $p_2(x)$ to determine local max/min



By design, i^{th} derivative of $p_k(x)$ matches i^{th} derivative of f,

$i = 1, \ldots, k$.

Goal: Do similar for $f: \mathbb{R}^n \to \mathbb{R}$ * ... figure out $p_k(\bar{x})$.

Recall: we've already done a 1^{st} order approximation:

matrix of 1^{st} partials ($1 \times n$ matrix)

Near $\bar{a}$, $f(\bar{x}) \approx f(\bar{a}) + Df(\bar{a})(\bar{x} - \bar{a})$

called this $h(\bar{x})$? Now, call it $p_1(\bar{x})$.

For 2^{nd} order approximation $p_2(\bar{x})$, need to keep track of 2^{nd} order partials:

Defn Sps $f: \mathbb{R}^n \to \mathbb{R}$. The <u>Hessian</u> of f at $\bar{a}$ is the $n \times n$ matrix:

$$Hf(\bar{a}) = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & & f_{x_2x_n} \\ & & & \\ f_{x_nx_1} & f_{x_nx_2} & & f_{x_nx_n} \end{bmatrix}$$

all eval. at $\bar{a}$

Note: If second order partials are cts, then mixed partials are equal so $Hf(\bar{a})$ is symmetric

$(Hf(\bar{a}))^T = Hf(\bar{a})$

Consider $\bar{x}-\bar{a}$: an n-vector $\begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix}$

*

$f: \mathbb{R} \to \mathbb{R}$

$f(\bar{a}) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$

Then $\frac{1}{2}(\bar{x}-\bar{a})^T Hf(\bar{a})(\bar{x}-\bar{a})$ is:

$$\frac{1}{2} \underbrace{\begin{bmatrix} (\bar{x}-\bar{a})^T \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} \\ Hf(\bar{a} \\ \\ \end{bmatrix}}_{n \times n} \underbrace{\begin{bmatrix} \\ \bar{x}-\bar{a} \\ \\ \end{bmatrix}}_{n \times 1} = \text{a Scalar}$$

Defn The second order Taylor polynomial of $f: \mathbb{R}^n \to \mathbb{R}$ at $\bar{a}$ is:

gives sum of all 2^{nd} partial information.

$$p_2(\bar{x}) = f(\bar{a}) + Df(\bar{a})(\bar{x}-a) + \frac{1}{2}\underbrace{(\bar{x}-\bar{a})^T Hf(\bar{a})(\bar{x}-\bar{a})}.$$

It's a better approximation of f than $p_1(\bar{x}) = f(\bar{a}) + Df(\bar{a})(\bar{x}-\bar{a})$.

↳ best quadric (as opp. to linear) approx. of f.