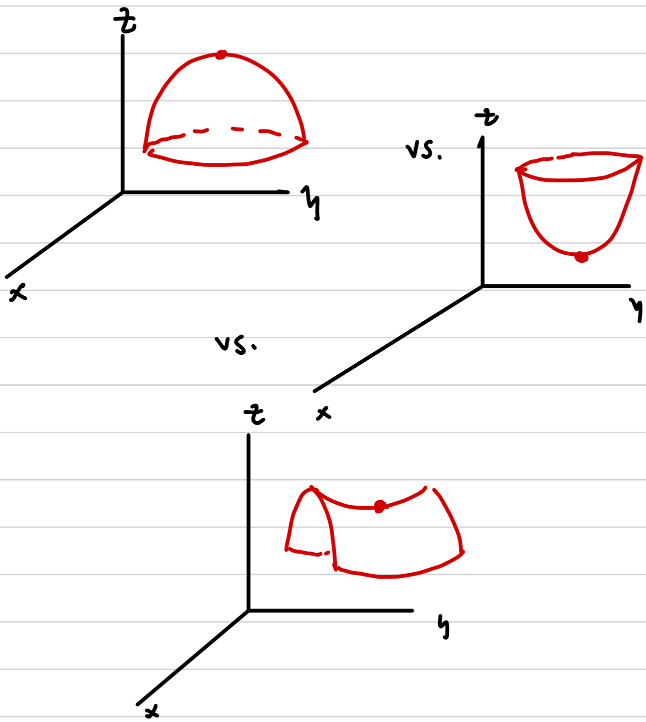
max/min

Extreme Values

Goal: Find and classify local max/local min/saddle pts

for $f: \mathbb{R}^n \to \mathbb{R}$.



Recall: $f: \mathbb{R} \to \mathbb{R}$, second derivative test:

1. find critical pts. where $f'(a) = 0$.

2. at each of these critical pts a, check $f''(a)$.

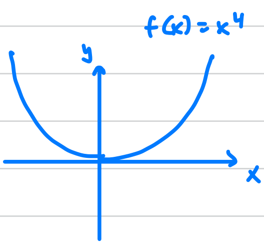
$$\overbrace{f(x) \approx f(a) + \underbrace{\cancel{f'(a)}}_{0}(x-a) + \frac{f''(a)}{2!}(x-a)^2}^{p_2(x)}$$

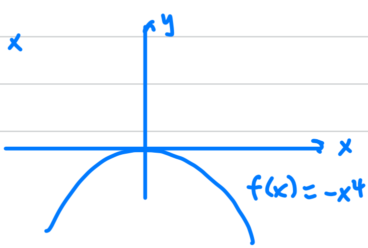
$\frac{(x-a)^2}{2!}$ → always positive

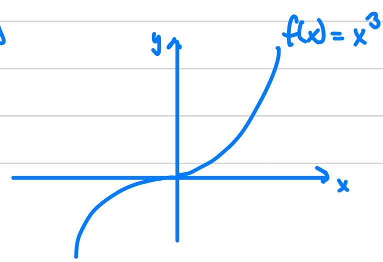
$f''(a) > 0 \Rightarrow$ local min at a

$f''(a) < 0 \Rightarrow$ local max at a

$f''(a) = 0$ .... don't know, not enough info. Find a diff method...







Now, for $f: \mathbb{R}^n \to \mathbb{R}$:

Thm Sps $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable. If f has a local extremum at $\bar{a}$, then $Df(\bar{a}) = [0 \cdots 0]$

zero matrix.

Note: not vice versa: $Df(\bar{a}) = \bar{0} \not\Rightarrow$ local max or min.

could have saddle pt.

proof: If max or min then $D_{\bar{v}} f(\bar{a}) = 0$ for all $\bar{v}$.

But $D_{\bar{v}} f(\bar{a}) = \nabla f(\bar{a}) \cdot \bar{v}$

Consider the partials:

$$0 = D_{\bar{e}_i} f(\bar{a}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_n}\right) \cdot \underbrace{(0, \ldots, 0, 1, 0, 0)}_{\bar{e}_i}$$

$$= \frac{\partial f}{\partial x_i}$$

So $\frac{\partial f}{\partial x_i} = 0$ for all i, but these are the components of $Df(\bar{a})$, so $Df(\bar{a}) = [0 \cdots 0]$.

So, if f is diff'ble,

1. Find critical points $\bar{a}$ where $Df(\bar{a}) = [0 \; \cdots \; 0]$
2. Classify each c.p. as local max, local min, or neither (saddle)

Sps $f: \mathbb{R}^n \to \mathbb{R}$ and $\bar{a}$ is a critical pt such that $Df(\bar{a}) = \boxed{0}$

Near $\bar{a}$,

$$f(\bar{x}) \approx f(\bar{a}) + \overset{0}{\cancel{Df(\bar{a})}}(\bar{x}-\bar{a}) + \underbrace{\overset{p_2(x)}{\tfrac{1}{2}(\bar{x}-\bar{a})^T Hf(\bar{a})(\bar{x}-\bar{a})}}_{\star}.$$

If $\star > 0$ for all $\bar{x} \neq \bar{a}$ near $\bar{a}$ (positive-definite), then

$f(\bar{x}) > f(\bar{a})$, so local min at $\bar{a}$.

If $\star < 0$ for all $\bar{x} \neq \bar{a}$ near $\bar{a}$ (negative definite), then

$f(\bar{x}) < f(\bar{a})$, so local max at $\bar{a}$

If $\det Hf(\bar{a}) \neq 0$ but $\star$ sometimes pos, sometimes neg, then saddle at $\bar{a}$.

If $\det Hf(\bar{a}) = 0$, not enough info.: try a diff method.