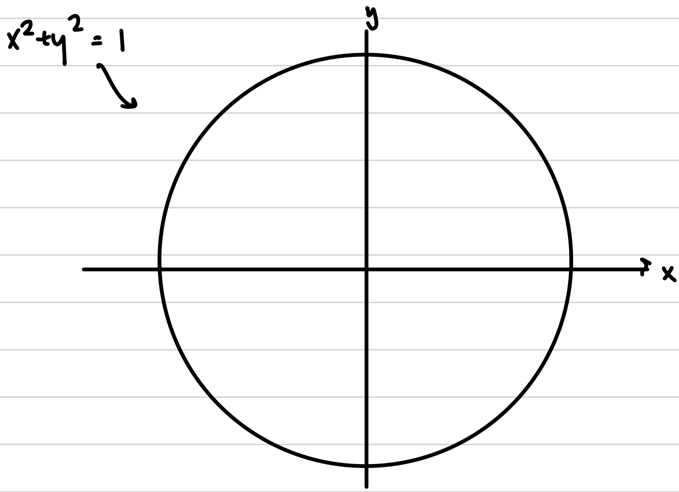
Lagrange Multipliers

A method for finding extrema of $f(x,y,z)$

<u>subject to a constraint</u> $g(x,y,z) = c$.

General Principle - via a 2-dim'l example.

Maximize $f(x,y) = y - x$ on circle $\underbrace{x^2+y^2}_{g(x,y)} = \underbrace{1}_{c}$

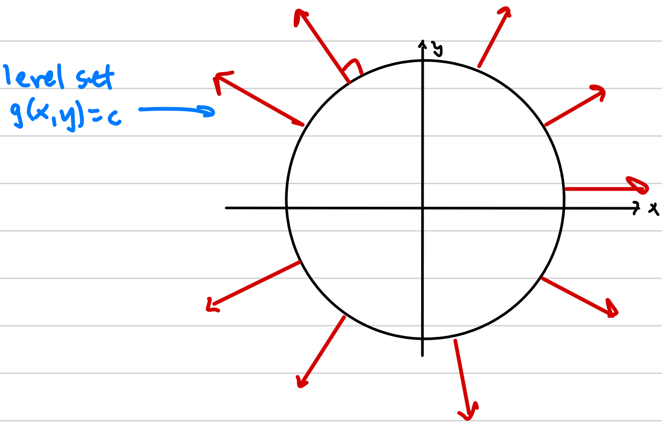


Here: $g(x,y) = x^2+y^2$

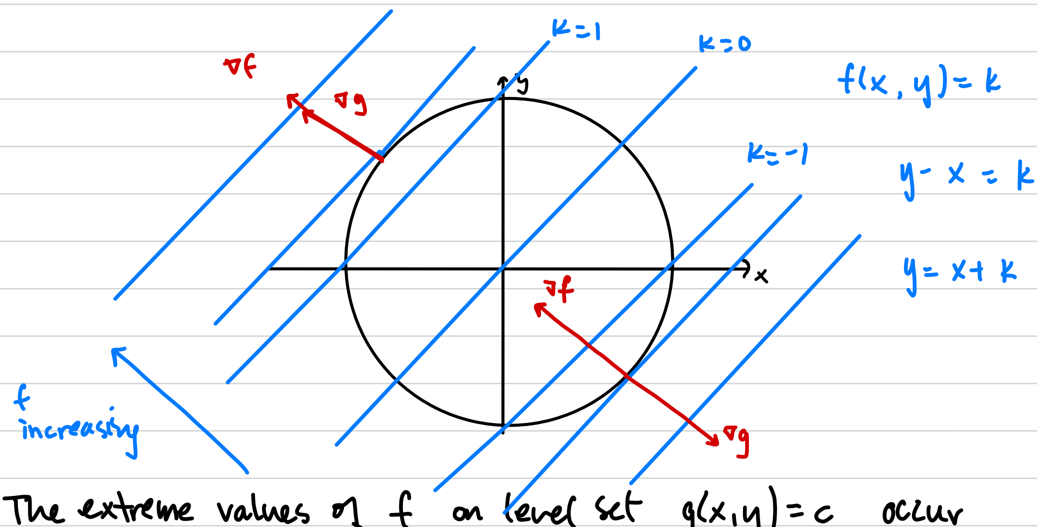
$f(x,y) = y - x$.

$\swarrow$ gradient is orthogonal to level set

Note: at any pt on circle, ∇g is $\perp$ to the circle.



Now, consider level sets of f.



The extreme values of f on level set $g(x,y)=c$ occur precisely when level set of f is tangent to level set of g.

This means ∇f is parallel to ∇g at extreme values.

lambda

$\nabla f = \lambda \nabla g$ for some λ.

Lagrange multiplier

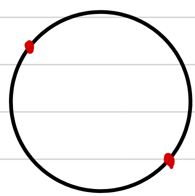
Here, where $f : \mathbb{R}^2 \to \mathbb{R}$, this gives three equations:

$f(x,y) = y - x$
$g(x,y) = x^2 + y^2$

$$\nabla f = \lambda \nabla g \begin{cases} -1 = 2x\lambda \\ 1 = 2y\lambda \end{cases} \quad ①$$

$$g(x,y) = c \ \{ \quad x^2 + y^2 = 1 \quad ②$$

solve for (x,y).



① $\Rightarrow$ $\lambda = -\dfrac{1}{2x} = \dfrac{1}{2y}$ $\quad y = -x$.

plug into ②

$y = -x$

$x^2 + (-x)^2 = 1 \rightsquigarrow 2x^2 = 1 \rightsquigarrow x = \pm \frac{1}{\sqrt{2}} \rightsquigarrow y = \mp \frac{1}{\sqrt{2}}$

so: c.p. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Note: here, circle is closed and bounded. By EVT, to

find absolute max/min: plug in and compare.

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \overset{y-x}{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

min

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

max. by EVT.

Summary!

- To find critical pts of $f: \mathbb{R}^n \to \mathbb{R}$ subject to constraint $g(x_1, x_2, \ldots, x_n) = c$, solve system:

level sets parallel → $\nabla f = \lambda \nabla g$ } n equations

constraint → $g(\bar{x}) = c$ } 1 equation

for $(x_1, \ldots, x_n)$ (and λ ... but ignore λ).

- Make an argument about which c.p. is max/min.