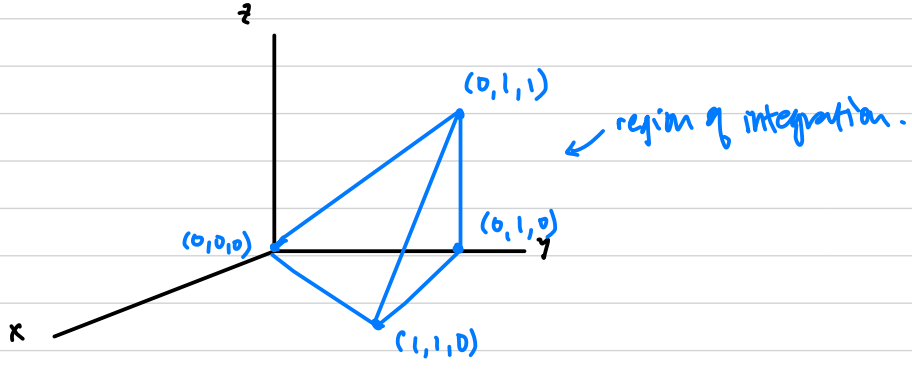
Ex Set up integral of $f(x,y,z) = xyz$ over tetrahedron with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$.

First:

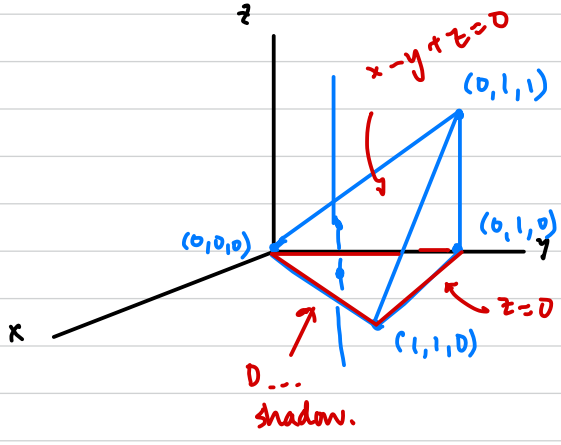


Want: $\iiint_B xyz \, dV$.

Let's integrate w.r.t. z first ⇒

project onto D in xy-plane

↳ get rid of z



upper boundary surface: plane $z = y - x$

lower boundary surface: xy-plane

↳ $z = 0$

Choose a point (x^*, y^*) in D, draw line through B parallel to z-axis to get z-limits of integration

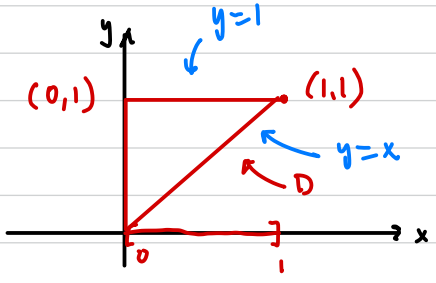
$$\int \int \left[\int_0^{y-x} xyz \, dz \right] dA$$

endpts ↑ curves ↑ z-values... represent surfaces

Ask: Do same surfaces work for all points in D?

(Here: yes. If not, repeat process.)

For the rest, set up a 2-dim'l integral over projection D in xy-plane:



$$\int_0^1 \int_x^1 \int_0^{y-x} xyz \, dz \, dy \, dx$$

or: $$\int_0^1 \int_0^y \int_0^{y-x} xyz \, dz \, dx \, dy$$

$$\int_0^1 \int_x^1 \frac{xyz^2}{2} \Big|_{z=0}^{z=y-x} dy \, dx$$

$$= \int_0^1 \int_x^1 \frac{xy(y-x)^2}{2} - 0 \; dy \, dx$$

$$= \text{etc.}$$