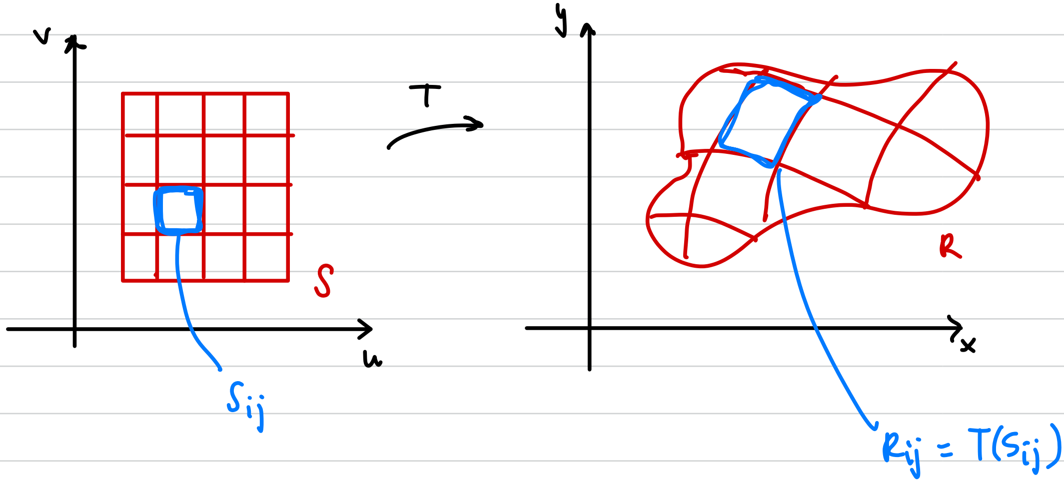

We have shown $\underbrace{\text{area } R_{ij}}_{\text{sub "rectangle" of } R} \approx |\det DT(u_0, v_0)| \overbrace{\Delta u \Delta v}^{\text{area } S_{ij}}$

But then:

from defn integral in xy-plane

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{f(x_{ij}^*, y_{ij}^*)}_{\text{equal}} \underbrace{\Delta A_{ij}}_{\text{approx equal, by above}}$$

$$\approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x(u_{ij}^*, v_{ij}^*), y(u_{ij}^*, v_{ij}^*)) \left|\det DT(u_{ij}^*, v_{ij}^*)\right| \Delta u \Delta v$$

$$= \iint_S \underbrace{f(x(u,v), y(u,v)) \, |\underbrace{\det DT(u,v)}_{\text{distortion factor}}|}_{\text{function being integrated.}} \, du\, dv$$

So:

Thm (Change of variables)

Sps T is a change of coordinates map with nonzero $DT(u,v)$ which maps a set S in the uv-plane to a set R in the xy-plane. Sps f is a continuous function. Then:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\,\underbrace{|\det DT(u,v)|}_{\text{distortion factor.}}\,du\,dv$$

↳ Note: this is higher dimensional version of u-substitution and trig-substitution.