We know:

If T is a change of coordinates transformation, then:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\underbrace{\left|\det DT(u,v)\right|}_{\text{distortion}}\,du\,dv$$

Now: apply to the polar change of coordinates

$$T(r,\theta) = (r\underbrace{\cos\theta}_{x(r,\theta)},\ r\underbrace{\sin\theta}_{y(r,\theta)})$$

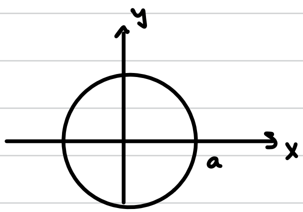
$$DT(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\left|\det DT(r,\theta)\right| = \left|r\cos^2\theta + r\sin^2\theta\right| = |r| = r \quad (r \geq 0)$$

Note: this distortion factor works for all polar double integrals.

Ex. Area of disk, radius a.

see motivating example



$\rightarrow \iint_D 1 \, dA$ ~ xy-integral

polar conversion

$= \int_0^{2\pi} \int_0^a 1 \, r \, dr \, d\theta$ *

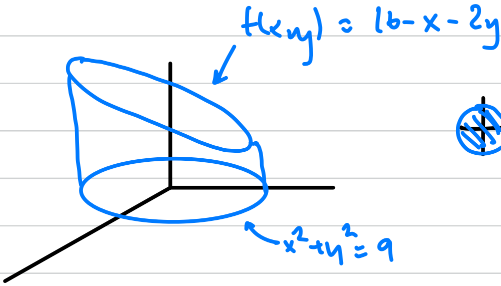
$= \int_0^{2\pi} \frac{r^2}{2} \Big|_0^a \, d\theta$

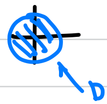
$= \int_0^{2\pi} \frac{a^2}{2} \, d\theta$

$= \frac{a^2}{2} \theta \Big|_0^{2\pi}$

$= \pi a^2.$

Ex. Set up integral that computes volume under plane $z = 16 - x - 2y$, above disk $x^2+y^2=9$.





$$\iint_D 16 - x - 2y \, dA$$ xy-integral

polar conversion

$$\int_0^{2\pi}\int_0^3 (16 - \underbrace{r\cos\theta}_{x(r,\theta)} - \underbrace{2r\sin\theta}_{y(r,\theta)})\, r \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 (16r - r^2\cos\theta - 2r^2\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} 8r^2 - \frac{r^3}{3}\cos\theta - \frac{2}{3}r^3\sin\theta \Big|_0^3 \, d\theta$$

$$= \int_0^{2\pi} 72 - 9\cos\theta - 18\sin\theta \, d\theta$$

$$= 72\theta - 9\sin\theta + 18\cos\theta \Big|_0^{2\pi}$$

$$= 144\pi$$