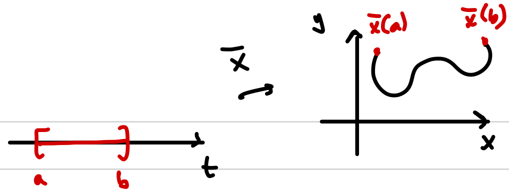


How to compute $\int_C f\,ds$?

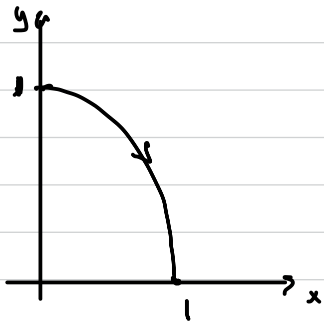
↳ Main idea: get everything in terms of parameter t.

(and, account for distortion that arises in the parametrization!)

We'll see: if C given by $\bar{x}(t) = (x(t), y(t))$, $a \le t \le b$,

$$\int_C f(x,y)\,ds = \int_a^b f(\bar{x}(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{\text{* distortion factor}}\,dt = \int_a^b f(\bar{x}(t))\,|\bar{x}'(t)|\,dt$$

Ex What is $\int_C x^3 y\,ds$ when C is the curve $y = 1 - x^2$ from $(0,1)$ to $(1,0)$?



Parametrize C: $\bar{x}(t) = (t, 1-t^2)$ $\quad 0 \le t \le 1$

Then we have:

$$\int_C x^3 y\,ds = \int_0^1 \underbrace{(t^3)}_{(x(t))^3}\underbrace{(1-t^2)}_{y(t)}\sqrt{(1)^2 + (-2t)^2}\,dt$$

↳ now, a single variable integral

* Idea: pull back along $\bar{x}$ to get everything in terms of t.

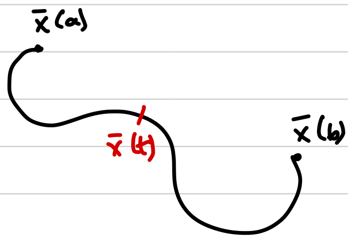
Why is this the right formula for $\int_C f(x,y)\,ds$?

Recall: arc length.

If C given by $\bar{x}(t) = (x(t), y(t))$, $a \le t \le b$,

$$L(C) = \int_a^b |\bar{x}'(t)|\,dt.$$

Define an arc length function:



variable here.

$$s(t) = \int_a^t |\bar{x}'(u)|\,du$$

gives length of C b/w $\bar{x}(a)$ and $\bar{x}(t)$

... increases as t increases.

By FTC:

variable

$$\frac{d}{dt}\int_a^t f(u)\,du = f(t).$$

$$\frac{ds}{dt} = \|\bar{x}'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \approx \frac{\Delta s}{\Delta t}$$

So $\Delta s_i \approx \|x'(t_i^*)\|\,\Delta t_i$

Thus

$$\int_C f(x,y)\,ds = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} \underbrace{f(x_i^*, y_i^*)}_{\text{equal}} \underbrace{\Delta s_i}_{\text{approx}}$$

$$= \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} f(\bar{x}(t_i^*))\, \|x'(t_i^*)\|\, \Delta t_i$$

$$= \int_a^b f(\bar{x}(t))\, \|\bar{x}'(t)\|\, dt.$$

So : again : to compute

- Parametrize C by $\bar{x}(t) = (x(t), y(t))$, $a \le t \le b$
- Then $\int_C f(x,y)\,ds = \int_a^b f(\bar{x}(t))\, \|\bar{x}'(t)\|\, dt.$

Remarks:

similar $\iint_A 1\, dA = \text{area}$

$\iiint_B 1\, dV = \text{volume}$

1. $\int_C 1\, ds$ gives length of C.

2. If $-C$ is same curve as C, traversed in opposite direction $\int_{-C} f\, ds = \int_C f\, ds$

b/c arc length doesn't depend on direction.