The Fundamental Theorem of Line Integrals

Recall: the gradient

grad : scalar-valued function $\to$ vector fields

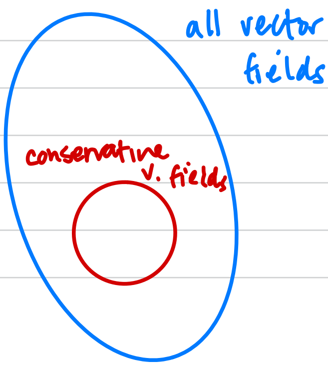
$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Recall: A vector field $\bar{F}$ is conservative if $\bar{F} = \nabla f$ for some scalar function f. In this case, f is called a potential function for $\bar{F}$.

Ex $\bar{F}(x,y) = (y, x)$ is conservative.

$\rightsquigarrow f(x,y) = xy$

$\nabla f = (y, x) = \bar{F}$ ✓



Nonex $\bar{F}(x,y) = (y, y^2)$ is **not** conservative.

Why not? If $f(x,y)$ existed, would have:

$$f_x = y \qquad f_y = y^2$$

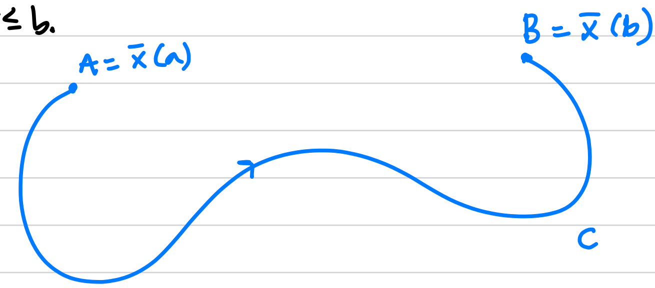
But: $(f_x)_y = 1 \qquad (f_y)_x = 0$

not equal... contradiction.

Thm (FTLI)

Sps. $\bar{F}$ is conservative, so $\bar{F} = \nabla f$ for some function f.

Let C be any path from A to B, parametrized by $\bar{x}(t)$, $a \le t \le b$.

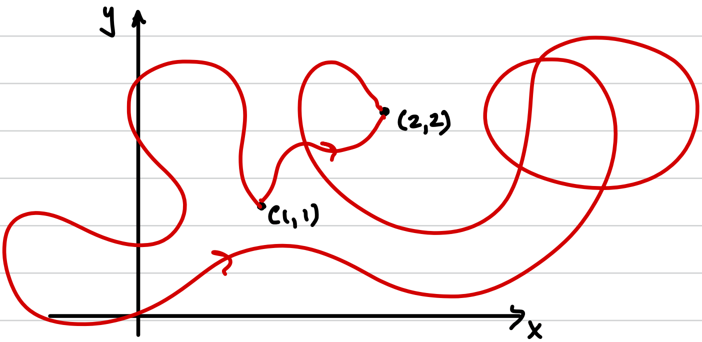


Then

$$\int_C \bar{F} \cdot d s = f(B) - f(A).$$

$\hookrightarrow f(\bar{x}(b)) - f(\bar{x}(a))$

Ex.



Let C be any curve from $(1,1)$ to $(2,2)$.

Let $\bar{F} = (y, x)$ (Recall $\bar{F} = \nabla f$ where $f(x,y) = xy$)

Then $\int_C \bar{F} \cdot d\bar{s} = f(2,2) - f(1,1) = 4 - 1 = 3.$

* Important consequence: If $\bar{F}$ is conservative ($\bar{F} = \nabla f$)

then $\int_C \bar{F} \cdot d\bar{s}$ depends only on value of f

at endpts A and B, not on C...

By contrast, for a general vector field: $\int_{C_1} \bar{F} \cdot ds \neq \int_{C_2} \bar{F} \cdot ds.$

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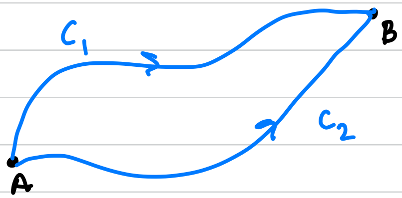
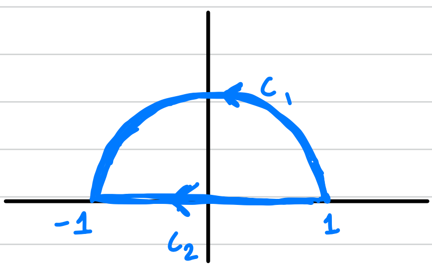


not conservative

Ex $\bar{F}(x,y) = (y, y^2)$

exercise:



$$\int_{C_1} \bar{F} \cdot d\bar{s} = -\frac{\pi}{2}$$

not equal.

$$\int_{C_2} \bar{F} \cdot d\bar{s} = 0$$