Another nice consequence of FTLI:

From $\int_C \nabla f \cdot d\bar{s} = f(B) - f(A)$ we know that

$\int_C \nabla f \cdot d\bar{s}$ is independent of path.

Turns out: $\int_C \bar{F} \cdot d\bar{s}$ indep of path

$\Leftrightarrow$

$\int_D \bar{F} \cdot d\bar{s} = 0$ for any closed path D.

Why?

Sps. $\int_C \bar{F} \cdot d\bar{s}$ indep of path. Let D be a closed path.

indep. of path

$$\int_D \bar{F} \cdot ds = \int_C \bar{F} \cdot ds$$

where C given by $\bar{x}(0) = A = B$.

$$= \int_0^0 F(\bar{x}) \cdot x'(t)\, dt$$

$$= 0.$$

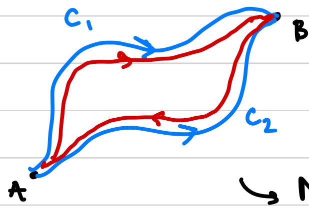
Alt: $\int_D \bar{F} \cdot ds = \int_D \nabla f \cdot ds$

$= f(B) - f(A) = f(A) - f(A) = 0$

OTOH, Sps $\oint_D \bar{F}\cdot d\bar{s} = 0$ for any closed path D.

WTS: $\int_{C_1} \bar{F}\cdot ds = \int_{C_2} \bar{F}\cdot ds$.

Consider



↪ Note $C = C_1 \cup -C_2$ is a closed path.

$$0 = \oint_C \bar{F}\cdot d\bar{s} = \int_{C_1} \bar{F}\cdot ds + \int_{-C_2} \bar{F}\cdot ds = \int_{C_1} \bar{F}\cdot ds - \int_{C_2} \bar{F}\cdot ds$$

$$\Rightarrow \int_{C_1} \bar{F}\cdot ds = \int_{C_2} \bar{F}\cdot ds.$$ ie independent of path.

Summary: On a simply connected region:

curl $\bar{F} = \bar{0}$ $\iff$ $\bar{F}$ conservative
(or $M_y = N_x$)

$\iff$ $\int_C \bar{F}\cdot d\bar{s}$ independent of path

$\iff$ $\oint_C \bar{F}\cdot d\bar{s} = 0$ for any closed path C.